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Applied Chaotic Dynamics: An Introduction

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APPLIED CHAOTIC DYNAMICS: AN INTRODUCTION

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Preface

These lecture notes served as the basis for a two credit graduate level seminar offered through the USU physics department during the fall quarter of 1989. It was oriented towards graduate students in physics and engineering and assumed no mathematical background beyond introductory differential equations. All problems were attempted by the students and discussed as a group. The seminar appears to have been successful in that the students enjoyed it and learned a lot.

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Chapter 1 - Qualitative/Historical Overview

In this introductory chapter various ways of visually presenting the evolution in time of dynamical systems are reviewed. This is followed by a qualitative review of types of system behavior from simple to complex.

A 'dynamical system' is a system which evolves in time. Thus each parameter can be viewed as a function of time (t). The most common ways of graphically representing the behavior of such systems are to plot various parameters (such as displacement) against time and to plot a trajectory in 'phase space' (or 'parameter space') which is traversed as time progresses.

One of the simplest examples to consider is a falling object. After it is released, it falls until it hits the ground. Aristotle attempted to explain this as the search of the object for its "natural place" at the center of the Earth. He hypothesized that the universe consisted of seven spheres with the Earth at the center. This led to a strengthened anthropomorphic view which was even more firmly established during the Middle Ages. Despite the philosophical implications of the later Copernican theory, and the revolutions in thought that accompanied it, modern civilization continues to owe much of its everyday view of the world to Aristotle. Figure 1 depicts the two representations of a falling object.

While Copernican theory revolutionized thought, the next type of motion to capture the attention of science was oscillatory motion, in the form of the pendulum. Galileo saw a measurable periodicity in the motion of a pendulum. Christian Huygens made use of this repetition to measure time. Figure 2 shows pendulum motion.

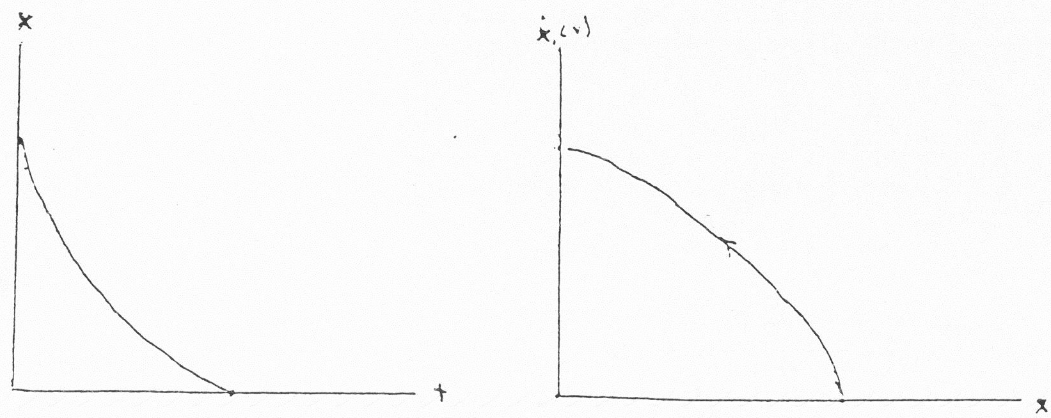


Figure 1 - Falling Object

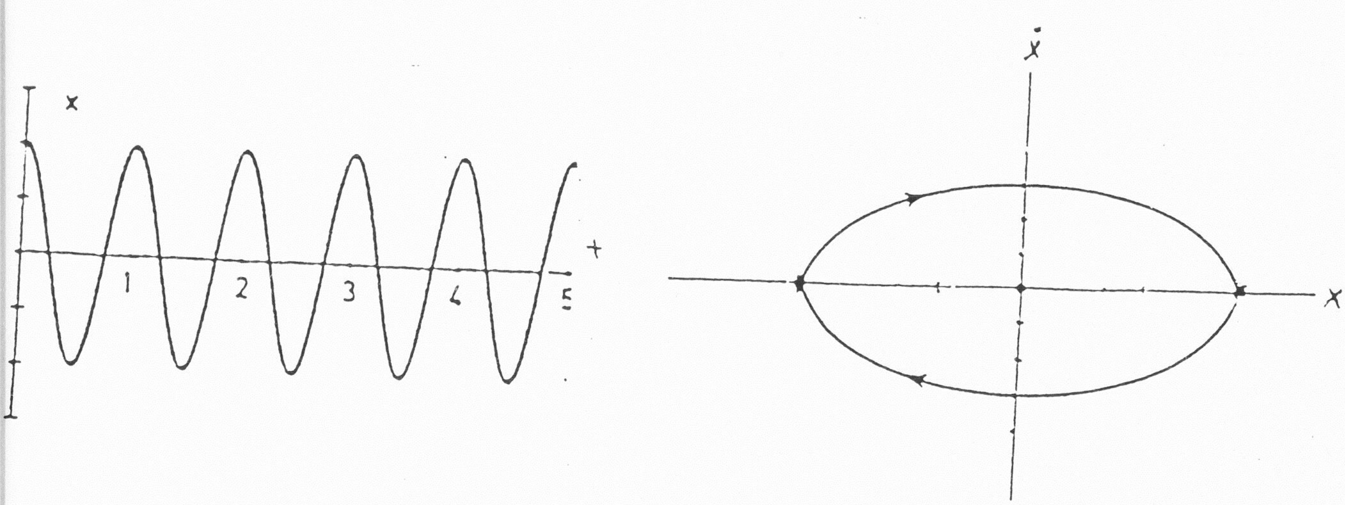


Figure 2 - Pendulum (Undamped)

An actual pendulum will not behave as shown in Figure 2. Friction and air resistance will damp the motion. This may explain in part why Aristotle thought that pendulum motion was still a manifestation of an object seeking its "natural place". He attributed the oscillations to the constraint of the rope. Ultimately, however, the bob comes to rest at its lowest point. Figure 3 shows the motion of a damped pendulum. Note that the volume enclosing the remaining trajectory becomes smaller as time passes and the system dissipates energy.

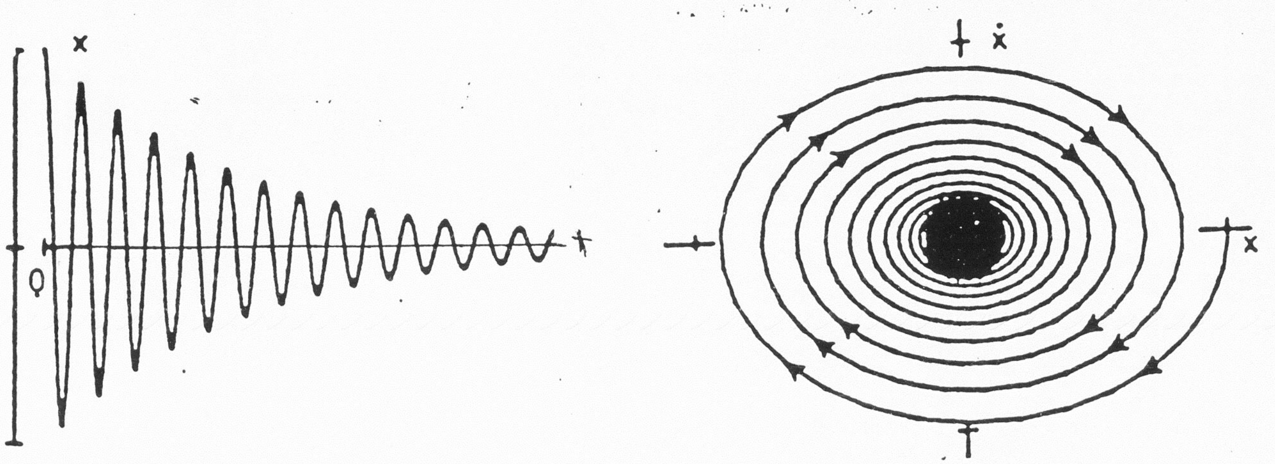


Figure 3 - Damped Pendulum

Galileo believed that the period of a pendulum was independent of the amplitude of its motion. Thus he believed that the motion was linear in nature. The linear pendulum equation is actually just an approximation, which becomes inadequate at larger amplitudes.

One of the first to see that things could begin to get exceedingly complicated was Jules Henri Poincaré. Poincaré investigated dynamical systems just before the turn of the century, with many of his published works appearing between 1880 and 1900. Many would call him the 'grandfather' of dynamical systems. Poincaré once wrote that:

"A very small cause which escapes our notice determines a considerable effect that we cannot fail to see, and then we say that the effect is due to chance. If we knew exactly the laws of nature and the situation of the universe at the initial moment, we could predict exactly the situation of that same universe at a succeeding moment. But even if it were the case that the natural laws had no longer any secret for us, we could still know the situation approximately. If that enabled us to predict the succeeding situation with the same approximation, that is all we require, and we should say that the phenomenon had been predicted, that it is governed by the laws. But it is not always so; it may happen that small differences in the initial conditions produce very great ones in the final phenomena. A small error in the former will produce an enormous error in the latter. Prediction becomes impossible..."

[Poincaré: Science and Method]

Poincarés' warning about this sensitive dependence on initial conditions was largely forgotten. Perhaps scientists were afraid to admit that their common linearized models, and the few simple nonlinear ones they understood, were woefully inadequate to explain many phenomena. Through the first half of this century they found their simple models to be adequate for their needs.

One day in the early 1960's a meteorologist named Edward Lorenz was using a system of three equations to model weather. Suddenly he was confronted with behavior of the type mentioned by Poincaré.

The simple models which had served science so well in the past were no longer adequate.

In the last thirty years, a growing number of scientists and mathematicians have studied systems exhibiting sensitive dependence on initial conditions. This behavior will be defined, loosely, as chaotic for now. There are different rigorous definitions of chaotic behavior, but sensitive dependence on initial conditions is a key characteristic of all chaotic systems. Figure 4 shows the type of behavior which may occur in chaotic systems.

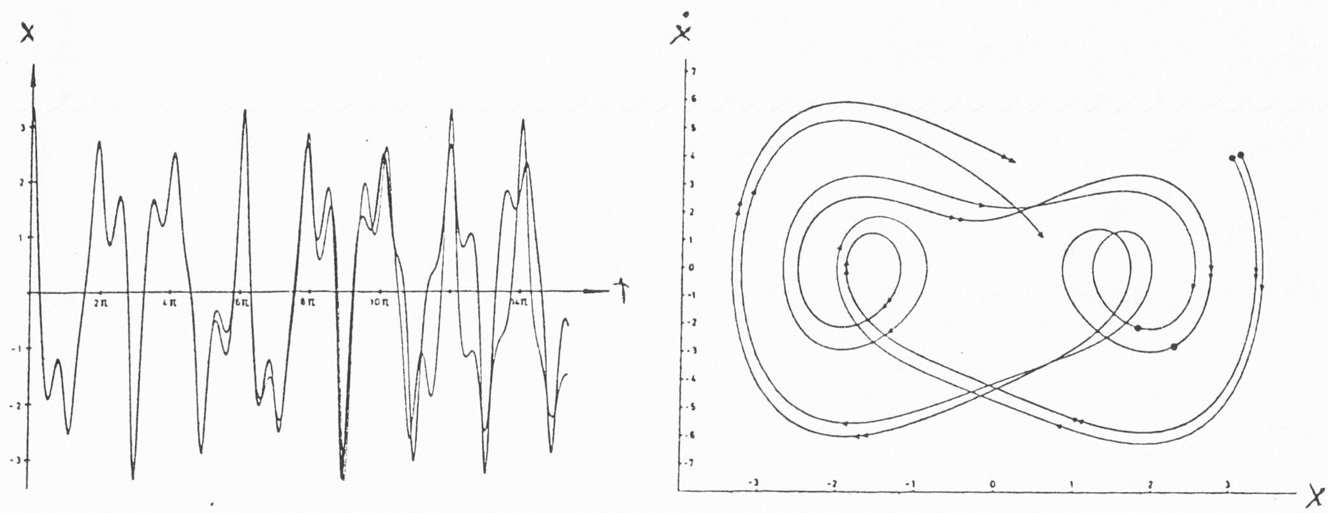


Figure 4 - Chaotic Behavior

Chaotic systems of equations have been used to model many types of observed behavior. Applications are diverse and include meteorology, traffic flow, structural mechanics, fluid mechanics, population biology, chemical reactions.

Further Reading

Interesting reading on the history of scientific thought from Aristotle through Einstein can be found in [Bohm; "Special Relativity"], particularly in the lengthy philosophical appendix. General background more specifically oriented towards the recent pioneers of chaotic dynamics abounds in [Gleick].

References to specific applications can be found in [Thompson and Stewart] and in the appendix to [Gleick].

Problems

1) Derive the equation of motion for the simple undamped pendulum [$x'' - (g/l)\sin x = 0$] and show that the equation can be approximated by [$x'' + (g/l)x = 0$].

2) The populations of two competing species can be modelled by the Lotka-Volterra equations:

$$x' = \beta_1 x(K_1 - x - \alpha_1 y)$$

$$y' = \beta_2 y(K_2 - y - \alpha_2 x)$$

where x is the population of prey and y is the population of predators.

(i) If the populations do not compete ($\alpha_1 = \alpha_2 = 0$), what do these equations reduce to?

(ii) Let $x' = y' = 0$ and find the equilibrium solutions (x_o, y_o). (There are four.)

(iii) Let $x = x_o + \delta x$ and $y = y_o + \delta y$. Assume δx and δy are small enough to ignore any of their products and find the linearized system of equations for small population changes in a neighborhood of the equilibrium point which lies in the first quadrant (off the axes).

3) In Figure 4 the trajectory appears to cross itself. Does it?

Chapter 2 - Some Necessary Advanced Calculus

In this chapter some basic definitions are given. Only the bare essentials required in the following chapters are presented. This should serve as a review for those who have had a course in advanced calculus and a brief introduction to some important concepts for those who have not.

Definition 2.1: Let F be a function defined on the set S. We say that F is <u>one-to-one</u> on S if and only if for every x and y in S $F(x) = F(y)$ implies $x = y$. If T is any set which contains F(S), then F is called a <u>mapping</u> from S to T, and we write $F: S \to T$. If $F(S) = T$, the mapping is said to be <u>onto</u> T.

Definition 2.2: A <u>metric space</u> is a non-empty set, M, of objects (called points) together with a function $d: MxM \to \mathbf{R}^1$, which is called the <u>metric</u> of the space. The metric d must satisfy the following for all points $\mathbf{x},\mathbf{y},\mathbf{z} \in M$:

(i) $d(\mathbf{x},\mathbf{x}) = 0$

(ii) $d(\mathbf{x},\mathbf{y}) > 0$ if $\mathbf{x} \neq \mathbf{y}$

(iii) $d(\mathbf{x},\mathbf{y}) = d(\mathbf{y},\mathbf{x})$

(iv) $d(\mathbf{x},\mathbf{y}) \leq d(\mathbf{x},\mathbf{z}) + d(\mathbf{z},\mathbf{y})$.

This metric space is denoted by (M,d_M) .

Example: Let M be any non-empty set, and $d(\mathbf{x},\mathbf{y}) = 0$ if $x = y$ and 1 if $x \neq y$. This is called the "discrete metric space". For additional examples of metrics, see Problems 1 and 6.

Definition 2.3: Let (S,d_s) and (T,d_T) be metric spaces and let $f: S \to T$. The function f is said to be continuous at a point **p** in S if for every $\epsilon > 0$ there is a $\delta > 0$ such that $d_T(f(\mathbf{x}),f(\mathbf{p})) < \epsilon$ whenever $d_S(\mathbf{x},\mathbf{p}) < \delta$.

Note that continuity is a local property. We say that f is continuous at **p**. If f is continuous at every point in a subset A of S, we say it is continuous on A.

Example: Let the function f be defined on [0,1] such that $f(x)= x$ if x is rational and $f(x) = 1 - x$ if x is irrational. Then f is continuous at $x = 1/2$. Let $p=1/2$ in definition 2.3, so $f(p)= 1/2$. Let $|x - p| < \delta$ and consider the two cases:

(i) x rational: $|f(x) - f(p)| = |x - p| < \delta$

(ii) x irrational: $|f(x) - f(p)| = |1 - x - p| = |p - x| < \delta$.

So for any given ϵ, let $\delta = \epsilon$ and the definition is satisfied (at $p = 1/2$).

However, f is not continuous anywhere else. Let $p \neq 1/2$ and let $0 < |x - p| < \delta$ and $\epsilon = |p - 1/2|$. For any δ, we can find x such that $|f(x) - f(p)| > |2p - 1| - \delta$. (Choose x so that either p or x is rational and the other is not.) Thus for any δ there is an x such that $|f(x) - f(p)| \geq |2p - 1| = 2|p - 1/2| = 2\epsilon$ so f is not continuous.

Definition 2.4: Let $f: S \to T$ be a function from the metric space (S,d_S) to (T,d_T). Let f be one-to-one (so that the inverse function f^{-1} exists) and onto. If f is continuous on S and f^{-1} is continuous on f(S) then f is called a homeomorphism, or a topological mapping. The metric spaces (S,d_S) and (T,d_T) are said to be homeomorphic. If, in addition, f and f^{-1} are C^r (r-times continuously differentiable), then f is called a C^r-diffeomorphism.

Definition 2.5: A sequence of points $\mathbf{x}_n$ in a metric space (S,d_S) is said to converge if there is a point $\mathbf{p}$ in S such that for every $\epsilon > 0$ there is an integer $N(\epsilon)$ such that $d(\mathbf{x}_n,\mathbf{p}) < \epsilon$ whenever $n \geq N$. This convergence is denoted by $\mathbf{x}_n \to \mathbf{p}$.

Definition 2.6: Let (S,d_S) be a metric space and let $A \subset S$. A point $\mathbf{x} \in S$ is called a limit point of A if there is a sequence (not containing $\mathbf{x}$) $\mathbf{x}_n$ in A such that $\mathbf{x}_n \to \mathbf{x}$. A set A is closed if it contains all its limit points. The union of a set and all its limit points is called the closure of A and is denoted by $\overline{A}$. A set A is open if its complement is closed.

Example: Let $\mathbf{x}_n$ be the sequence in $\mathbf{R}^1$ such that $x_n = 1/n$. Then $x_n \to 0$. The set $A = (0,1]$ is not closed because it does not contain zero, which is a limit point. $\overline{A} = [0,1]$.

Definition 2.7: A subset U of S is dense in S if $\overline{U} = S$.

Example: The set of rational numbers is dense in **R**. To show this, it must be shown that every number is a limit point of the set of rationals. It suffices to show that the positive rationals which are less than one form a dense subset of [0,1]. Let x be an irrational number in this interval. To construct a sequence x_n of rationals such that $x_n \to x$ we let $x_n = p_n/n$ where $0 < p_n < n$ and choose p_n such that $|x_n - x|$ is a minimum.

Definition 2.8: A set of points Λ is a Cantor set if it is closed and contains no intervals, and if every point is a limit point of other points in the set.

Example: The classic introductory example is called the "Cantor middle thirds set", Λ_3. Start with a whole interval, such as [0,1]. Remove the middle third, (1/3,2/3). Next remove the middle third of each remaining interval, (1/9,2/9) and (7/9,8/9). Continue inductively this process of removing open sets.

Note that in the nth step we remove 2^n intervals, and that each of these has length $1/3^{n+1}$. (The first step is n=0.) Thus after the nth step the total length removed is $\sum(2^i/3^{i+1}) = (1/3)\sum(2/3)^i$, where the sum is from i = 0 to i = n. Letting $n \to \infty$ and summing the resulting geometric series to get

$$(1/3)\sum(2/3)^i = (1/3)[1 - 2/3]^{-1} = (1/3)(3) = 1 .$$

Thus Λ_3 has zero length.

It can now be shown that Λ_3 meets the criteria for a Cantor set. Since each interval removed is open, the union of everything removed is open. The complement of this is Λ_3 and is closed. The set cannot contain any intervals since the length would then be non-zero.

To see that each point is a limit point, consider that at any step the endpoints of all remaining intervals are elements of Λ_3. Choose any point p in Λ_3. At the nth step, p is contained in some remaining interval. Choose x_n to be an endpoint of this interval, but not equal to p. Then $x_n \to p$.

This concludes our brief review of basic general mathematics. In the next chapter some additional mathematics is developed which is oriented more specifically towards dynamical systems.

Further Reading

The reader who is interested in continued study of chaotic dynamics is strongly encouraged to enroll in or to attend a course on Advanced Calculus. There are many references available on this material. One of the best is [Apostol], although the uninitiated may have some trouble with the level of rigor.

Problems

1) The open ball $B(\mathbf{a},r)$ is defined as the set of all points $\mathbf{x}$ such that $d(\mathbf{x},\mathbf{a}) < r$. Sketch $B(\mathbf{0},1)$ in $(\mathbf{R}^2,d_i)$ for the following metrics. [$\mathbf{x} = (x_1,x_2)$].

$d_1(\mathbf{x},\mathbf{a}) = [(x_1 - a_1)^2 + (x_2 - a_2)^2]^{1/2}$ (Euclidean)

$d_2(\mathbf{x},\mathbf{a}) = \max_{i=1,2} |x_i - a_i|$

$d_3(\mathbf{x},\mathbf{a}) = \sum |x_i - a_i|$ (sum is over i)

2) Let the function f be defined on [0,1] by f(x) = 1 if x is rational and f(x) = 0 if x is irrational. Show that f is not continuous anywhere.

3) Which of the following are homeomorphisms and/or (C^1-) diffeomorphisms on their domain of definition?

i) sinh x

ii) cosh x

iii) x^n ; $n \in \mathbf{Z}^+$

iv) $J_o(x)$

4) Find all limit points:

i) $\{x|\ \sin(1/x) = 0\}$

ii) $\{x|\ x = (1/n) + (1/m);\ n,m \in \mathbf{Z}^+\}$

iii) $\{x|\ x = 2^{-n} + 5^{-m};\ n,m \in \mathbf{Z}^+\}$

5) Starting with a square with sides one unit long, divide each side into fifths. Remove the middle horizontal and vertical strips, leaving four squares with sides of length (2/5). Show that repeating this process leaves behind a set of points which is a Cantor set. (Hint: Find the total area remaining after the nth step and show that it goes to zero as $n \to \infty$.)

6) Let $(\sum_2, d)$ be the metric space $\{\mathbf{s}=(s_0 s_1 s_2 \ldots) \mid s_i = 0 \text{ or } 1\}$ and let $d(\mathbf{s},\mathbf{t}) = \sum |s_i - t_i|/2^i$ where the sum is over all i. Thus the space consists of all binary sequences.

i) Show that d is a metric on $\sum_2$.

ii) Let $\mathbf{s}_1$ = (001 001 001 ...)

$\mathbf{s}_2$ = (010 110 110 ...)

$\mathbf{s}_3$ = (001 110 110 ...)

$\mathbf{s}_4$ = (001 010 110 ...)

Find $d(\mathbf{s}_1, \mathbf{s}_i)$ for i = 2,3,4. What does it mean for two sequences to be "close"?

iii) Find the maximum distance M between two points. Give two examples of points $\mathbf{s},\mathbf{t}$ such that $d(\mathbf{s},\mathbf{t}) = M$.

iv) Let $Per(\sigma)$ be the set of all repeating sequences $\mathbf{s} = (s_1 \ldots s_n\ s_1 \ldots s_n\ \ldots)$. Show that $Per(\sigma)$ is dense in $\sum_2$. (Hint: For $\mathbf{t} \in \sum_2$ construct a sequence τ_n of points in $Per(\sigma)$ such that $\tau_n \to \mathbf{t}$ as $n \to \infty$.)

Chapter 3 - An Introduction to Iterated Maps

In this chapter some of the basic features of iterated maps, and of dynamical systems in general, are introduced. The map $x_{n+1} = x_n + c$ will serve as a basis for much of the discussion.

Definition 3.1: The composition of two functions $f(\mathbf{x})$ and $g(\mathbf{x})$, denoted by $f{\circ}g(\mathbf{x})$, is defined by $f{\circ}g(\mathbf{x}) = f(g(\mathbf{x}))$. The nth iterate of a function F is denoted by $F^n(\mathbf{x})$ and is defined inductively as $F^n(\mathbf{x}) = F{\circ}F^{n-1}(\mathbf{x})$.

Note that F^n does not indicate any type of exponential. To avoid confusion with differentiation, the nth derivative will sometimes be indicated by $F^{(n)}$.

If the chain rule for derivatives is applied to the nth iterate $h(\mathbf{x}) = F^n(\mathbf{x})$ it results in the following expression:

$$h'(\mathbf{x}) = F'(F^{n-1}(\mathbf{x})) \cdot F'(F^{n-2}(\mathbf{x})) \cdots F'(\mathbf{x}).$$

Definition 3.2: Let (S,d_s) be a metric space and let F be a function on S, $F: S \to S$. The orbit of a point $\mathbf{p}$ in S is the set of points $\mathbf{p}, F(\mathbf{p}), F^2(\mathbf{p}), \cdots$.

Definition 3.3: A point $\mathbf{x}$ is a fixed point of the function f if $f(\mathbf{x}) = \mathbf{x}$. It is a periodic point with period n if $f^n(\mathbf{x}) = \mathbf{x}$. The smallest n for which $f^n(\mathbf{x}) = \mathbf{x}$ is called the prime period of $\mathbf{x}$. The set of all iterates of a periodic point form a periodic orbit.

The set of periodic points of period (not necessarily prime) n is denoted by $Per_n(f)$, and the set of fixed points by Fix(f).

Definition 3.4: A point $\mathbf{x}$ is an <u>eventually periodic (fixed)</u> point of period n if it is not periodic (fixed) itself, but there exists $m > 0$ such that $f^{n+i}(\mathbf{x}) = f^i(\mathbf{x})$ for all $i \geq m$.

Note that a point $\mathbf{x}$ is eventually periodic if, for some i, $\mathbf{x}_o = f^i(\mathbf{x})$ is periodic.

Example: Let $F: \mathbf{R} \to \mathbf{R}$, $F(x) = x^2$. Then $F^n(x) \to \infty$ if $|x| > 1$, $F^n(x) \to 0$ if $|x| < 1$, $F^n(1) = 1$ for all n, and $F^n(-1) = 1$ if $n \geq 1$. Thus 1 is a fixed point and -1 is an eventually fixed point. All other points approach (are <u>asymptotic</u> to) either 0 or ∞.

Example: 1 is an eventually periodic point of $F(x) = x^2 - 1$.

Example: Let S^1 denote the unit circle in the plane, and let the metric on S^1 be the length of the shortest arc connecting two points. A point on the circle may be specified by its angle θ in radians. The same point θ is specified by $\theta + 2\pi k$ for any integer k. Let $f: S^1 \to S^1$ be defined by $f(\theta) = 2\theta$. (In the complex plane we could write $f(z) = z^2$.) Since $f^n(\theta) = 2^n\theta$, $\theta \in Per_n(f)$ if and only if $2^n\theta = \theta + 2\pi k$, or if $\theta = 2\pi k/(2^n - 1)$. Thus the set $Per_n(f)$ consists of the $(2^n - 1)$th roots of 1. The set of all periodic points of f is dense in S^1. To show this, construct a sequence of points with ω, arbitrary, as a limit point as follows. Let the nth entry in the sequence be the element of $Per_n(f)$ which is closest to ω, i.e. the closest $(2^n - 1)$th root of 1. Each element of this sequence will be a periodic point of f, and $|x_n - \omega| \to 0$ as $n \to \infty$. Note that $f(0) = 0$, so $\theta = 0$ is a fixed point of f. If $\theta = 2\pi k/2^n$ then $f^n(\theta) = 2\pi k$ and θ is eventually fixed. Thus the set of eventually fixed points of f is also dense in S^1.

Theorem 3.1: A homeomorphism cannot have eventually periodic points.

Proof: Assume f is a homeomorphism and $\mathbf{x}$ is an eventually periodic point of f. Then $f^{i+n}(\mathbf{x}) = f^i(\mathbf{x})$ for some i. Since F is a homeomorphism, it is one-to-one, so $f(\mathbf{s}) = f(\mathbf{t})$ if and only if $\mathbf{s} = \mathbf{t}$. Thus $f^{i+n-1}(\mathbf{x}) = f^{i-1}(\mathbf{x})$. If this argument is repeated i-1 more times, the result is that $f^n(\mathbf{x}) = f(\mathbf{x})$. This shows that $\mathbf{x}$ itself is periodic, so it is not eventually periodic (Defn. 3.4).

Theorem 3.2: A homeomorphism of $\mathbf{R}$ cannot have periodic points with prime period greater than two.

Example: There are homeomorphisms of $\mathbf{R}$ with periodic points of prime period two. Let $f_1(x) = -x$ and $f_2(x) = -x^3$. Then $Per_2(f_1)$ contains all points. All of these except the origin have a prime period of two. $Per_2(f_2)$ consists of the points -1, 1, and 0.

A useful technique in the study of systems in $\mathbf{R}^1$ is <u>graphical analysis</u>. The function f is graphed and the diagonal $(y = x)$ is

included on the graph. To follow the orbit of a point p, start on the diagonal at (p,p) and draw a vertical line to (p,f(p)). Next follow a horizontal line back to the diagonal, (f(p),f(p)). Another vertical line will intersect the graph of f at $(f(p),f^2(p))$. Going back to the diagonal leads to $(f^2(p),f^2(p))$. Repeating this process displays the orbit on the diagonal.
See Fig. 3.1.

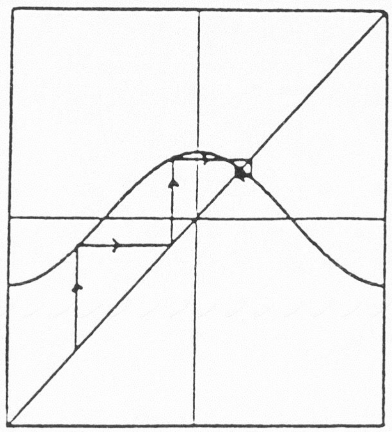


Figure 3.1 - Graphical Analysis of cos x

Fixed points are those points where f crosses the diagonal. By considering sections of graphs near fixed points, it is easy to see that orbits of nearby points will be attracted to a fixed point p if $|f'(p)| < 1$ and will be repelled if $|f'(p)| > 1$.

Example: Let $F_c(x) = x^2 + c$ and consider $F_0(x)$. The fixed points are 0 and 1, and -1 is eventually fixed. $F_0'(x) = 2x$.
$|F_0'(0)| = 0 < 1$, so 0 is attracting. $|F_0'(1)| = 2 > 1$, so 1 is repelling. This is easily verified through the use of graphical analysis.

The above observations can be extended to periodic points. Note that if $\mathbf{x} \in Per_n(f)$, then $\mathbf{x}$ is a fixed point of f^n.

Definition 3.5: Let **p** be contained in $Per_n(f)$. If $|(f^n)'(\mathbf{p})| < 1$, **p** is called an <u>attracting periodic point</u>, an <u>attractor</u>, or a <u>sink</u>. If $|(f^n)'(\mathbf{p})| > 1$, **p** is called a <u>repelling periodic point</u>, a <u>repellor</u>, or a <u>source</u>. Attractors and repellors are called <u>hyperbolic</u>. The point **p** is <u>nonhyperbolic</u> if $|(f^n)'(\mathbf{p})| = 1$.

Example: Let $F_c(x) = x^2 + c$, as before. For $c > 1/4$ the graph of F_c will lie above the diagonal and all orbits will diverge to $+\infty$. If $c = 1/4$, a nonhyperbolic fixed point appears at $x = 1/2$, where $F_{1/4}(x)$ is tangent to the diagonal. When $-3/4 < x < 1/4$ there are two fixed points, the greater one being repelling and the smaller attracting. This splitting of the one nonhyperbolic point into two hyperbolic points fixed points is an example of a <u>bifurcation</u>. Nonhyperbolic points are often associated with bifurcations.

Next, jump ahead and consider $F_{-2}(x)$. While changes occur between $c = -3/4$ and $c = -2$, they will be discussed later. The fixed points are the solutions to $x = x^2 - 2$, or 2 and -1. Thus all interesting dynamics occur on the interval $I = [-2,2]$. Points outside this interval have orbits diverging to $+\infty$. The graph of $F_{-2}(x)$ on the interval of interest is shown in Figure 3.2.

Note that $[-2,0]$ and $[0,2]$ are each mapped onto $[-2,2]$, or that F_{-2} folds I over itself twice. A plot of $F_{-2}^2(x)$ shows that it folds

I over itself four times, and a plot of $F_{-2}^3(x)$ shows that this process repeats itself. See Figure 3.3.

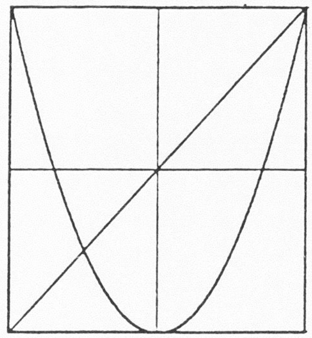


Figure 3.2 - Graph of $F_{-2}(x) = x^2 - 2$

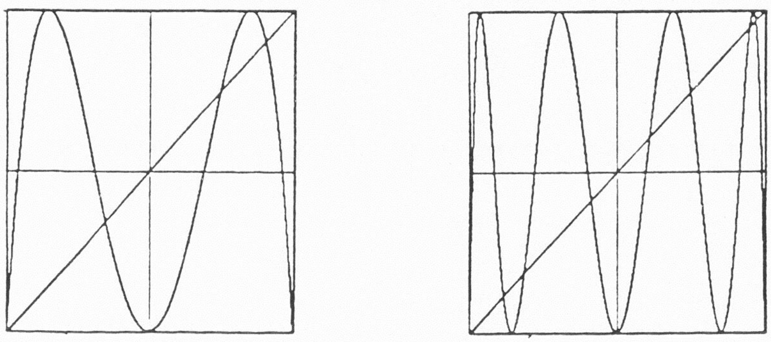


Figure 3.3 - Graphs of $F_{-2}^2(x)$ and $F_{-2}^3(x)$

From Figure 3.3 it is clear that $F_{-2}^2(x)$ has four fixed points and that F_{-2}^3 has eight. Thus there are four elements in $Per_2(F_{-2})$ and eight in $Per_3(F_{-2})$. In general, $Per_n(F_{-2})$ contains 2^n points.

Theorem 3.3 (Sarkovskii): Order the positive integers in the following manner:

$$3 \to 5 \to 7 \to \ldots \to 2\cdot 3 \to 2\cdot 5 \to \ldots \to 2^2\cdot 3 \to 2^2\cdot 5 \to \ldots \to 2^3 \to 2^2 \to 2 \to 1.$$

Assume $F: \mathbf{R} \to \mathbf{R}$ is continuous and has a periodic point of prime period n. If k follows n in the above ordering, then F also has a periodic point of prime period k.

Corollary 3.3.1: If F has a point of prime period three, then F has prime periodic points with all periods.

Corollary 3.3.2: If F has a periodic point whose period is not a power of two, then F has infinitely many periodic points.

Despite the powerful results this theorem assumes very little of F, requiring only continuity. It is very counter-intuitive. After all, how hard should it be to set up a system with a period three oscillation and no others?

In 1975 James Yorke and a student of his, Tien-Yien Li, published a proof of Corollary 3.3.1 in a famous paper titled 'Period Three Implies Chaos'. This was the birth of the term 'chaos'. Only several years later at a conference in East Berlin did Yorke discover that his result was just a special case of a Theorem published by Sarkovskii in 1964 in the 'Ukrainian Mathematics Journal'.

The question which arises naturally at this point is "Where are all these orbits?". They are generally not apparent when the function is iterated on a calculator or a computer. This question will be addressed, to some extent, in the following chapters. In the next chapter some methods of analysis which allow us to better understand the dynamics of such maps are developed.

Further Reading

Much of the material in this chapter is covered in [Devaney] and in the article by Devaney in [Keen]. [Devaney] includes some interesting additional examples of maps of the circle.

For a proof of Sarkovskii's Theorem, the reader is referred to [Devaney] or to the original article [Sarkovskii].

Problems

1) Describe the dynamics of $F_{-1}(x)$. Find all fixed and periodic points and determine whether they are attracting, repelling, or non-hyperbolic.

2) Show that for $-3/4 < c < 1/4$ F_c has two fixed points, one attractor and one repellor.

Chapter 4 - Analysis of the Logistic Map

In this chapter the iterated map $x_{n+1} = \mu x_n(1 - x_n)$ is analyzed through the use of symbolic dynamics and the concept of topological conjugacy is introduced. This map arises in studies of populations. (See Appendix A, Problem 2).

Let $F_\mu(x) = \mu x(1 - x)$. For $\mu > 0$, the basic features include two fixed points at 0 and at $p = (\mu - 1)/\mu$. $F_\mu(1) = 0$, so 1 is eventually fixed. If $x \notin I = [0,1]$, then the orbit of x diverges to $-\infty$. See Figure 4.1.

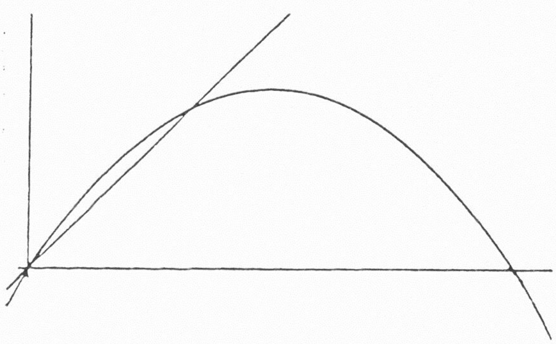


Figure 4.1 - Graph of $F_\mu(x)$, $\mu>1$

If $\mu \in (1,3)$, p is attracting and 0 is repelling. All non-diverging orbits are attracted to p. This can be verified graphically. At $\mu = 3$, p is non-hyperbolic and a bifurcation occurs.

Jump ahead now to $\mu > 4$, and let $F_{\mu>4}(x)$ denote $F_\mu(x)$ for any $\mu > 4$ for the remainder of this chapter. For $\mu = 4$ the maximum value of $F_4(x)$ is 1, and when $\mu > 4$ a section of the graph rises above $F_{\mu>4}(x) = 1$. Thus points in this section escape from the interval I after only one iteration. See Figure 4.2.

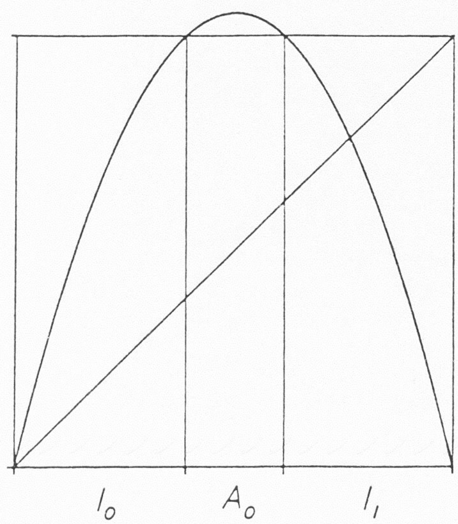


Figure 4.2 - $F_\mu(x)$ for $\mu > 4$

Let A_0 denote the set of points in I which escape on the first iteration. A_0 is an open interval centered at 1/2. It is open since its endpoints are mapped to 1, and centered at 1/2 by the symmetry of the function. Thus $A_0 = (1/2 - \Delta, 1/2 + \Delta)$ where Δ will depend on μ.

Next note that the two sections of I remaining if A_0 is removed, call them I_0 and I_1, are each mapped onto [0,1] (refer to Fig. 4.2). Therefore, each contains a subinterval which is mapped into $(1/2 - \Delta,\ 1/2 + \Delta)$. If a point x in I_0 or I_1 is mapped into this interval, then $F_{\mu>4}{}^2(x)$ will escape from I. Call the union of these two subintervals A_1.

Since the orbits of points in A_0 or A_1 escape from I and diverge to $-\infty$, the set of points with interesting (i.e. bounded) orbits is a subset of $I - (A_0 \cup A_1)$. See Figure 4.3.

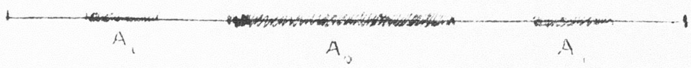


Figure 4.3 : $I - (A_0 \cup A_1)$

At this point four intervals are left. However, each of these is mapped onto I_0 or I_1. (The left interval is mapped onto I_0, for instance.) Since points in a certain subinterval of I_0 or of I_1 are mapped out of I after two iterations, there is a subinterval of each of the remaining four intervals in which the orbits which diverge. Call the union of these four intervals A_2.

This process can be continued and the set of points remaining (i.e. those with interesting orbits) is $I - (\cup A_n)$ where the union is over all integers n. This procedure of repeatedly removing open subintervals is similar to the construction of the Cantor set Λ_3 (see Chapter 2). Indeed, the set of interesting points is a Cantor set, which will be denoted Λ_μ.

If $F_{\mu>4}^{\ 2}$ and $F_{\mu>4}^{\ 3}$ are graphed, the same type of folding that occurred with $f = x^2 - 2$ can be seen (see Chapter 3), except that some of the 'fold' spills over from I each time. Since periodic points of $F(x)$ of period i are fixed points of $F^i(x)$, this repeated folding shows that F^n has 2^n fixed points, or that $Per_n(F)$ consists of 2^n elements. See Figure 4.4.

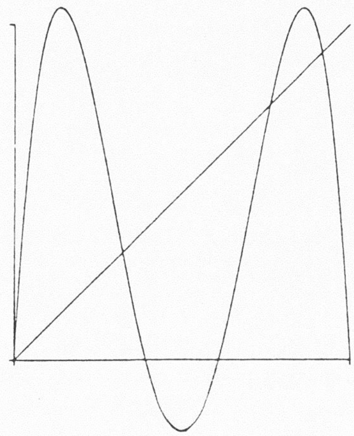


Figure 4.4 - Graph of $F_{\mu>4}{}^2(x)$

While $F_\mu(x)$ for $\mu < 3$ was simple, the function $F_{\mu>4}(x)$ is not. What happens between $\mu = 3$ and $\mu = 4$ will be touched on later. Further analysis of F(x) in the Euclidean space $\mathbf{R}^1$ would appear to be extremely difficult at best. The next task, therefore, is to develop an equivalent model in a different metric space.

Definition 4.1: Let the sequence space Σ_n consist of all infinite sequences of integers between 0 and n-1 (inclusive). Let the metric $d_{\Sigma n}$ define the distance between points in Σ_n and be given by

$$d_{\Sigma n}(\mathbf{s},\mathbf{t}) = \Sigma \; |s_i - t_i|/n^i \quad .$$

where the sum is over all i. In problem 6 in chapter 2 it was shown that $d_{\Sigma 2}$ is a metric on Σ_2.

Definition 4.2: The shift map σ: $\Sigma_n \rightarrow \Sigma_n$ is the function $\sigma(\mathbf{s}) = \sigma(s_0 s_1 s_2 \ldots) = (s_1 s_2 s_3 \ldots)$. That is, σ simply drops the first entry in the sequence.

Theorem 4.1: Let $\mathbf{s},\mathbf{t} \in \Sigma_2$, and suppose $s_i = t_i$ for $i \leq n$. Then $d_{\Sigma 2}(\mathbf{s},\mathbf{t}) \leq 1/2^n$. Conversely, if $d_{\Sigma 2}(\mathbf{s},\mathbf{t}) \leq 1/2^n$, then $s_i = t_i$ for $i \leq n$.

Proof: First assume $s_i = t_i$ for $i \leq n$. Then $d_{\Sigma 2}(\mathbf{s},\mathbf{t}) = \Sigma|s_i - t_i|/2^i$ where the sum is over all $i > n$. Thus $d_{\Sigma 2}(\mathbf{s},\mathbf{t}) \leq 2 - \Sigma(1/2^i)$ where this sum is over all $i \leq n$, or $d_{\Sigma 2}(\mathbf{s},\mathbf{t}) \leq 2 - (2 - (1/2)^n)$. This leads directly to $d_{\Sigma 2}(\mathbf{s},\mathbf{t}) \leq 1/2^n$. Now assume $s_j \neq t_j$ for some $j \leq n$. Then $d_{\Sigma 2}(\mathbf{s},\mathbf{t}) \geq 1/2^j \geq 1/2^n$. So if $d_{\Sigma 2}(\mathbf{s},\mathbf{t}) < 1/2^n$, $s_i = t_i$ for all $i \leq n$. ■

Theorem 4.2: The shift map σ on Σ_2 is a continuous function.

Proof: For any $\epsilon > 0$, some $\delta > 0$ must be found such that $d_{\Sigma 2}(\sigma(\mathbf{s}),\sigma(\mathbf{t})) < \epsilon$ whenever $d_{\Sigma 2}(\mathbf{s},\mathbf{t}) < \delta$. For any $\epsilon > 0$, choose n such that $1/2^n < \epsilon$. Let $\delta = 1/2^{n+1}$. If $d_{\Sigma 2}(\mathbf{s},\mathbf{t}) < \delta$, $s_i = t_i$ for $i \leq n+1$, by Theorem 4.1. That means $[\sigma(\mathbf{s})]_i = [\sigma(\mathbf{t})]_i$ for $i \leq n$. Using Theorem 4.1 again, $d_{\Sigma 2}[\sigma(\mathbf{s}),\sigma(\mathbf{t})] \leq 1/2^n < \epsilon$. ■

What is needed now is some way of relating σ on Σ_2 to $F_{\Sigma>4}$ on Λ_μ.

Definition 4.3: The <u>itinerary function</u> $S(x): \Lambda_\mu \to \Sigma_2$ is the sequence $S(x) = s_0 s_1 s_2 \ldots$ where $s_j = 1$ if $F_{\mu>4}{}^j(x) \in I_1$ and $s_j = 0$ if $F_{\mu>4}{}^j(x) \in I_0$. (Refer to Figure 4.2).

Theorem 4.3: $S(x): \Lambda_\mu \to \Sigma_2$ is a homeomorphism.

Recall that a homeomorphism is also called a topological mapping. What Theorem 4.3 says is that Λ_μ and Σ_2 are the same as sets.

Theorem 4.4: $S \circ F_{\mu>4} = \sigma \circ S$.

Theorem 4.4 says that taking $F_{\mu>4}(x)$ (in Euclidean space) and then $S(F_{\mu>4}(x))$ yields the same sequence as taking $S(x)$ and then applying the shift map to it. The dynamics of $F_{\mu>4}(x)$ on Λ_μ and the dynamics of $\sigma(\mathbf{s})$ on Σ_2 are equivalent! Homeomorphisms such as $S:\Lambda_\mu \to \Sigma_2$ are important and deserve a special title.

Definition 4.4: Let $f: S \to S$ and $g: T \to T$ be two maps on the metric spaces (S,d_s) and (T,d_T). f and g are called <u>topologically conjugate</u> if there is a homeomorphism $h: S \to T$ such that $h \circ f = g \circ h$. A function such as h is called a <u>topological conjugacy</u>.

For $F_{\mu>4}$, σ, and S, the following diagram holds:

$$\begin{array}{ccc} \Lambda_\mu & \xrightarrow{F} & \Lambda_\mu \\ S\updownarrow & & S\updownarrow \\ \Sigma_2 & \xrightarrow{\sigma} & \Sigma_2 \end{array}$$

The homeomorphism S gives a one-to-one correspondence between fixed points, periodic points, and all other features of F and σ.

The following properties of $\sigma(\mathbf{s})$ can be established:

1) $Per_n(\sigma)$ has 2^n elements
2) $Per(\sigma)$ is dense in Σ_2
3) σ has a dense orbit in Σ_2.

From the topological conjugacy of $F_{\mu>4}$ and σ it follows immediately that:

1) $Per_n(F_{\mu>4})$ has 2^n elements
2) $Per(F_{\mu>4})$ is dense in Λ_μ
3) $F_{\mu>4}$ has a dense orbit in Λ_μ.

While the first property could be deduced graphically, the other two are not obvious. Topological conjugacy has made it a simple matter to establish these facts, since they are readily apparent in $(\Sigma_2, d_{\Sigma 2})$. Other properties can also be established, but this requires more mathematics than has been developed here. In the next chapter we will give a definition of chaotic behavior and look at some additional examples.

Further Reading

The proofs for several key results have been omitted here, primarily because their mathematical complexity exceeds that which has been developed or is being assumed. They can all be found in [Devaney]. The problems in chapter 1.6 of [Devaney] establish some properties concerning non-wandering orbits and recurrence, but also require additional mathematical background. At this point, the reader who has not studied advanced calculus should have ample motivation to do so!

Problems

1) Show that $\sigma(\mathbf{s})$ has the three properties listed. [Hints: For property 2, refer to problem 6 in chapter 2. For property 3 construct a sequence of all 1 digit binary numbers (0,1) followed by all 2 digit binary numbers (0,1,2,3) etc.]

2) Is $\sigma(\mathbf{s})$ a homeomorphism? Give two reasons why or why not.

3) Let $F_c(x) = x^2 + c$, and $F_\mu(x) = \mu x(1 - x)$. Show that if $c < 1/4$ there is a unique $\mu > 1$ such that $F_c(x)$ is topologically conjugate to $F_\mu(x)$. [Hint: Try $h(x) = \alpha x + \beta$.]

4) Let $G_n(x) = 4x(1 - x) + (1+\epsilon)\sin n\pi x$, and consider $G_3(x)$. Graph $G_3(x)$. Sketch $G_3^2(x)$. [Hint: Reason graphically, do **not** try to explicitly find $G_3(G_3(x))$.] Describe the set of points with bounded orbits. What would you expect the appropriate topologically conjugate space to be? Give the conjugacy, list the the properties which could be deduced, and write down a dense orbit for G_3 in its conjugate space.

Chapter 5 - Chaotic Iterated Maps

The maps studied in the last two chapters, $F_c(x)$ and $F_\mu(x)$, exhibited very complicated behavior for certain ranges of the parameters c and μ. Such behavior can be called 'chaotic'. In this chapter 'chaos' is defined formally, and examples of chaotic maps are introduced. The definition introduced here is that given in [Devaney] and is topological in nature. Other definitions are possible, some requiring a knowledge of measure theory, ergodic theory, and/or other subjects.

Definition 5.1: Let (S,d_s) be a metric space. A map $f:S \to S$ is called <u>topologically transitive</u> if for any pair of open sets $A,B \subset S$ there is a positive integer k such that $f^k(A) \cap B \neq \emptyset$.

Under iteration, a topologically transitive function maps any region of S to any other so that S cannot be partitioned into disjoint sets such that each is mapped into itself.

Example: Let (S^1,d) be the unit circle with a 'shortest arc length' metric. Define the <u>translation map</u> $T_\lambda:S \to S$ by $T_\lambda(\theta) = \theta + 2\pi\lambda$. Let λ be irrational, $\lambda \notin \mathbf{Q}$. If $T_\lambda^n(\theta) = T_\lambda^m(\theta)$, then $(n - m)\lambda$ is an integer and, since $\lambda \notin \mathbf{Q}$, n = m. Thus the orbit of θ consists of distinct points. An infinite set of points on the circle must have a limit point.[1] Therefore, for any $\epsilon > 0$

[1] The existence of a limit point is guaranteed by the Bolzano-Weierstrass Theorem.

there are integers n and m such that $|T_\lambda{}^n(\theta) - T_\lambda{}^m(\theta)| < \epsilon$.[2] If $k = n - m$, $d(T_\lambda{}^k(\theta),\theta) < \epsilon$. Note that T_λ preserves length, so $T_\lambda{}^k$ maps the arc between θ and $T_\lambda{}^k(\theta)$ to the arc connecting $T_\lambda{}^k(\theta)$ and $T_\lambda{}^{2k}(\theta)$. Thus for any ϵ an integer $p(\epsilon)$ can be found such that iterations of $T_\lambda{}^p$ partition S^1 into arcs of length less than ϵ. Therefore, the orbit of any point θ comes arbitrarily close to any other point in S^1. Thus T_λ is topologically transitive for irrational λ.

Definition 5.2: Let $f:S \to S$ be defined on the metric space (S,d_s). The map f has <u>sensitive dependence on initial conditions</u> if there is a $\delta > 0$ such that for every $\mathbf{x} \in S$ and any neighborhood N of $\mathbf{x}$ there exists at least one $\mathbf{y} \in N$ such that $d_S(f^n(\mathbf{x}),f^n(\mathbf{y})) > \delta$ for some n. If there is a δ such that this condition holds for **every** pair $\mathbf{x},\mathbf{y} \in S$, f is called <u>expansive</u>.

Sensitive dependence means that the orbit of at least one point arbitrarily close to $\mathbf{x}$ separates from the orbit of $\mathbf{x}$. Expansiveness means that the orbits of all nearby points diverge.

Definition 5.3: A map $f:S \to S$ on (S,d_s) is called <u>chaotic</u> if it has sensitive dependence on initial conditions, is topologically transitive, and if Per(f) is dense in S.

Example: $F_\mu(x) = \mu x(1 - x)$ is chaotic on Λ_μ for $\mu > 4$. Let δ be any number less than the length of A_0, where A_0 was the first

---

[2] Every convergent sequence is a Cauchy sequence.

interval removed and separated I_0 and I_1. If $x,y \in \Lambda_\mu$, and $x \neq y$, then they have different itineraries, so $S(x) \neq S(y)$. Recall that $S(x)$ maps Λ_μ onto Σ, so $S(x) \neq S(y)$ means that the sequences differ in at least one entry, say the nth. But this means that either $F_\mu^n(x)$ or $F_\mu^n(y)$ is in I_0 and the other is in I_1, so $|F_\mu^n(x)-F_\mu^n(y)|>\delta$. Thus F_μ exhibits sensitive dependence on initial conditions. Since F_μ has a dense orbit in Λ_μ, it is topologically transitive. It was shown earlier that the periodic points are dense.

Example: $f: S^1 \to S^1$, $f(\theta) = n\theta$ is chaotic. Since angular distance between two points is multiplied by n each iteration, f has sensitive dependence on initial conditions. In fact, it is expansive. Since any small arc in S^1 expands under iteration and eventually covers S^1 entirely, the map is topologically transitive. That periodic points are dense was shown for n=2 in chapter 3. A similar argument to the one presented there holds for other n as well.

Example: The Tchebycheff polynomials $T_n(x)$ are solutions to the differential equation $(1 - x^2)y'' - xy' + n^2y = 0$, and can be defined by $T_n(x) = \cos(n \arccos x)$. The first four are:

$$T_0(x) = 1$$

$$T_1(x) = x$$

$$T_2(x) = 2x^2 - 1$$

$$T_3(x) = 4x^3 - 3x \ .$$

$T_n(x)$ is chaotic on $I = [-1,1]$ for all n. Let $g_n:S^1 \to S^1$, $g_n(\theta)=n\theta$. Let $T_n(x):I \to I$ be the nth Tchebycheff polynomial. Let $h:S^1 \to I$,

$h(\theta) = \cos\theta$, so h is the projection of S^1 onto I. Now $h \circ g_n = \cos n\theta$ and $T_n \circ h = \cos(n \arccos(\cos\theta)) = \cos n\theta$. Figure 5.1 summarizes the situation.

$$
\begin{tikzcd}
S^1 \arrow[r, dashed, "g_n"] \arrow[d, "h"'] & S^1 \arrow[d, "h"] \\
I \arrow[r, dashed, "T_n"] & I
\end{tikzcd}
$$

Figure 5.1

Note that $h(\theta)$ is not one-to-one, so we do not have a topological conjugacy between g_n and T_n. In this case, the two maps are said to be semi-conjugate. This semi-conjugacy can be used to show that $T_n(x)$ is chaotic on I.

To see that $T_n(x)$ is topologically transitive, let U and V be two arbitrary open intervals in I. Then there exists two intervals u and v in S^1 which project onto U and V. $g_n^k(u) \cap v \neq \emptyset$ for some k, since $g_n(\theta)$ is topologically transitive. Taking projections shows that $T_n^k(U) \cap V \neq \emptyset$.

Similarly, any neighborhood U of an arbitrary point x in I is the projection of some interval u in S^1. Since $g_n(\theta)$ is expansive, $S^1 \subset g_n^k(u)$ for some k. Projecting back to I, $I \subset T_n^k(U)$. Thus, there are points in U which separate by at least 1 ($\delta = 1$).

Finally, for any interval u in S^1 there is a periodic point x of $g_n(\theta)$ in u since $Per(g_n)$ is dense in S^1. Periodic points of $g_n(\theta)$ project to periodic points of $T_n(x)$, so any neighborhood U of I contains a periodic point of $T_n(x)$ and $Per(T_n)$ is dense in I.

So far only one dimensional iterated maps have been considered. Continuous maps can often be reduced to iterated ones through the use of Poincaré maps. One dimensional maps sometimes offer insight into the behavior of higher dimensional maps with parameters in a certain range. The two dimensional Hénon map, for instance, reduces in one special case to the quadratic map $F_c(x)$.

In the next chapter, some additional mathematical tools are introduced which will be necessary for the study of dynamical systems which are continuous and/or higher dimensional.

Further Reading

Most of the material in this chapter can be found in [Devaney], including a specific case of a single Tchebycheff polynomial. The generalization presented here is an extension of that to all T_n. Some additional information on the iterated two dimensional Hénon map and its reduction to the quadratic map is presented in [Thompson/Stewart].

Problems

1) For translations of the circle show that all fixed points of T_λ are fixed points if λ is rational, $\lambda = p/q$. (Hint: Consider T_λ^q.) Is T_λ a chaotic map for either case, $\lambda \in \mathbf{Q}$ or $\lambda \notin \mathbf{Q}$? Give two reasons in each case.

2) Verify that $y = T_n(x) = \cos(n \arccos x)$ is a solution to the equation $(1 - x^2)y'' - xy' + n^2y = 0$.

3) Show that a periodic point θ_p of $g_n(\theta) = n\theta$ projects under $h(\theta) = \cos\theta$ to a periodic point of $T_n(x)$.

4) The Hénon map is given by:

$$x_{n+1} = 1 - ax_n^2 + y_n$$

$$y_{n+1} = bx_n \ .$$

Show that if b = 0 and x is scaled, this map reduces to

$$\bar{t}_{n+1} = \bar{t}_n - a.$$

5) Like most special functions, the Tchebycheff polynomials have a recursion relation: $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$.

a) Find $T_4(x)$ from $T_3(x)$ and $T_2(x)$ (given in the chapter).

b) Using the polynomial form of $T_4(x)$, show that $h \circ g_4 = T_4 \circ h$ where $h = \cos\theta$ and $g_4 = 4\theta$. (Hint: Look up the multiple angle formula for $\cos 4\theta$.)

6) The first two Hermite polynomials are $H_0 = 1$ and $H_1 = 2x$. The recursion relation is $H_{n+1} = 2xH_n - 2nH_{n-1}$. Find $H_2(x)$ and show that it is chaotic on some interval.

Chapter 6 - More Mathematical Basics

In this chapter, some of the tools necessary in the study of higher dimensional systems and systems modelled by ordinary differential equations are reviewed.

Definition 6.1: Let $\mathbf{f}:\mathbf{R}^n \to \mathbf{R}^n$, $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_n(\mathbf{x}))$ where, for each i, $f_i(\mathbf{x}):\mathbf{R}^n \to \mathbf{R}^1$. $\mathbf{f}(\mathbf{x})$ is said to be differentiable at a point $\mathbf{x}_0$ if there exists a linear transformation $\mathbf{L}:\mathbf{R}^n \to \mathbf{R}^n$ (i.e. an n x n matrix) such that $\mathbf{f}(\mathbf{x}_0 + \mathbf{h}) = \mathbf{f}(\mathbf{x}_0) + \mathbf{L}(\mathbf{h}) + \|\mathbf{h}\|\mathbf{E}(\mathbf{h})$ where $\mathbf{E}_c(\mathbf{h}) \to 0$ as $\|\mathbf{h}\| \to 0$.

Definition 6.2: The matrix representation of the linear transformation $\mathbf{L}(\mathbf{h})$ in definition 6.1 has the form

$$\mathbf{F}'(\mathbf{x}_0) = \begin{bmatrix} \partial F_1/\partial x_1 & \partial F_1/\partial x_2 & \cdots & \partial F_1/\partial x_n \\ \vdots & & & \vdots \\ \partial F_n/\partial x_1 & \partial F_n/\partial x_2 & \cdots & \partial F_n/\partial x_n \end{bmatrix},$$

where all partial derivatives are evaluated at $\mathbf{x}_0$. The determinant of this matrix is called the Jacobian determinant and is denoted by $J_F(\mathbf{x}_0)$.

Theorem 6.1 (Chain Rule): Let $\mathbf{f}(\mathbf{x})$ be continuously differentiable at $\mathbf{x}_0$ and let $\mathbf{g}(\mathbf{x})$ be continuously differentiable at $\mathbf{f}(\mathbf{x}_0)$, where $\mathbf{f}$ and $\mathbf{g}$ are both functions from $\mathbf{R}^n$ to $\mathbf{R}^n$. If $\mathbf{g}\circ\mathbf{f}$ is defined on an open set U such that $\mathbf{x}_0 \in U$, then $\mathbf{g}\circ\mathbf{f}$ is continuously differentiable at $\mathbf{x}_0$ and $(\mathbf{g}\circ\mathbf{f})'(\mathbf{x}_0) = \mathbf{g}'(\mathbf{f}(\mathbf{x}_0))\cdot\mathbf{f}'(\mathbf{x}_0)$.

Example: Let $\mathbf{f}(u,v) = (u \cos v,\ u \sin v)$ and

$\mathbf{g}(x,y) = ((x^2 + y^2)^{1/2},\ \arctan(y/x))$ and assume

$(u,v) \neq (0,0)$ and $(x,y) \neq (0,0)$.

Then
$$\mathbf{f}'(u,v) = \begin{bmatrix} \cos v & -u \sin v \\ \sin v & u \cos v \end{bmatrix}$$

and
$$\mathbf{g}'(x,y) = \begin{bmatrix} x/(x^2+y^2)^{1/2} & y/(x^2+y^2)^{1/2} \\ x/(x^2+y^2) & y/(x^2+y^2) \end{bmatrix}.$$

To apply the chain rule, it is necessary to evaluate $\mathbf{g}'$ at $f(u,v)$:

$$\mathbf{g}'(u \cos v,\ u \sin v) = \begin{bmatrix} \cos v & -u \sin v \\ \sin v & u \cos v \end{bmatrix}.$$

The chain rule is now used, and the two matrices are multiplied:

$$(\mathbf{g}\circ\mathbf{f})'(u,v) = \mathbf{g}'(u \cos v,\ u \sin v)\cdot\mathbf{f}'(u,v) = I_2$$

where I_2 is the 2 x 2 identity matrix.

Note that the chain rule is a matrix product. In this example, $(\mathbf{g}\circ\mathbf{f})' = I_2$ because $(\mathbf{g}\circ\mathbf{f})(u,v) = (u,v)$. The function $\mathbf{f}$ is a transformation from polar to cartesian coordinates and $\mathbf{g}$ is a transformation from cartesian back to polar coordinates.

As the next example shows, the Jacobian determinant of a transformation indicates how volume scales.

Example: Consider the ellipsoid $(x/5)^2 + (y/3)^2 + (z/2)^2 = 1$. Let $x = 5u$, $y = 3v$, and $z = 2w$, or $(x,y,z) = \mathbf{T}(u,v,w) = (5u,3v,2w)$. Then the ellipsoid in (u,v,w) coordinates is $u^2 + v^2 + w^2 = 1$, or a unit sphere. The volume of a unit sphere is $(4/3)\pi$. Now,

$\mathbf{T}'(u,v,w) = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, or $J_T(u,v,w) = 5\cdot 3\cdot 2 = 30$. The volume of the original ellipsoid is $V = (4/3)\pi \cdot J_T = 40\pi$. This is easily verified by a direct calculation, $V = (4/3)\pi(a)(b)(c) = (4/3)\pi(5)(3)(2) = 40\pi$. This case is somewhat trivial, since all that was done was to scale the coordinates. However, this procedure could be used in cases where the ellipsoid was translated and/or rotated. In the more general case, the derivative matrix would not be diagonal.

Theorem 6.2 (Inverse Function Theorem): Let $\mathbf{f}:\mathbf{R}^n \to \mathbf{R}^n$ be continuously differentiable on an open set U such that $\mathbf{x}_0 \in U$ and let $J_f(\mathbf{x}_0) \neq 0$. Then there is an open set V such that $\mathbf{x}_0 \in V$ and $\mathbf{f}$ has a continuously differentiable function $\mathbf{f}^{-1}$ defined on $\mathbf{f}(V)$.

Example: Consider a one-dimensional case, $f:\mathbf{R}^1 \to \mathbf{R}^1$. Let $f(x) = x^n$ where n is a positive integer. Then $f'(x) = nx^{n-1}$, and $J_f = f'$. Since $J_f(x) \neq 0$ for $x \neq 0$, $f^{-1}(x)$ can be defined on a neighborhood of any point other than zero. Note that this conclusion is strictly <u>local</u> in nature, since $f^{-1}(x)$ is not globally unique when n is even. At $x = 0$ the theorem is inconclusive and other means must be used to establish the existence or non-existence of a local inverse. If n is even, f is not one-to-one in any interval containing the origin and the inverse will not exist. However, when n is odd the origin is a point of inflection and not an extrema and f^{-1} does exist.

Example: Let $\mathbf{f}(r,\theta) = (r \cos \theta, r \sin \theta)$. Then $J_f(r,\theta) = r$, so f^{-1} will exist everywhere with the exception of the origin. f^{-1} changes cartesian to polar coordinates. At (0,0), θ is not unique.

Example: The Hénon map is defined by

$$x_{n+1} = 1 - ax_n^2 + y_n$$

$$y_{n+1} = bx_n \ .$$

$H'(x,y) = \begin{bmatrix} -2ax & 1 \\ b & 0 \end{bmatrix}$, and $J_H = -b$. Thus the Hénon map is invertible everywhere unless $b = 0$. At $b = 0$ it reduces to the quadratic map, which is not invertible. The price paid for stepping down to one dimension is that the system cannot be followed backwards in time. That is, x_n cannot be found as a unique function of x_{n+1} when $b = 0$.

The definition of differentiability is easily extended to functions $\mathbf{f}:\mathbf{R}^n \to \mathbf{R}^m$. In the general case of $n \neq m$ the derivative is a rectangular matrix, so its determinant is not defined.

Definition 6.3: Let $\mathbf{f}:\mathbf{R}^n \to \mathbf{R}^m$, $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}),\ldots,f_m(\mathbf{x}))$, $\mathbf{x} \in \mathbf{R}^n$. The function $\mathbf{f}(\mathbf{x})$ is said to be <u>differentiable</u> at $\mathbf{x}_0 \in \mathbf{R}^n$ if there is a linear transformation $\mathbf{L}:\mathbf{R}^n \to \mathbf{R}^m$ so that $\mathbf{E}_c(\mathbf{h}) \to 0$ as $\|\mathbf{h}\| \to 0$ and $\mathbf{f}(\mathbf{x}_0+\mathbf{h}) = \mathbf{f}(\mathbf{x}_0) + \mathbf{L}(\mathbf{h}) + \|\mathbf{h}\|\mathbf{E}_c(\mathbf{h})$. The transformation $\mathbf{L}$ is an m x n matrix and is called the <u>Fréchet derivative</u> of $\mathbf{f}$ at $\mathbf{x}_0$. It has the form

$$\mathbf{f}'(\mathbf{x}_0) = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 & \cdots & \partial f_1/\partial x_n \\ \vdots & & & \vdots \\ \partial f_m/\partial x_1 & \partial f_m/\partial x_2 & \cdots & \partial f_m/\partial x_n \end{bmatrix} ,$$

where the partial derivatives are evaluated at $\mathbf{x}_0$.

If $\mathbf{f}:\mathbf{R}^n \to \mathbf{R}$, the Fréchet derivative is a row vector, which is the <u>gradient</u> of $\mathbf{f}$. If $\mathbf{f}:\mathbf{R}^n \to \mathbf{R}^n$, the Fréchet derivative is an n x n matrix and reduces to the case first introduced. The chain rule can also be extended to the general case. Instead of square matrices, rectangular ones are multiplied.

Definition 6.4: Let $\mathbf{F}:\mathbf{R}^{n+m} \to \mathbf{R}^m$, $\mathbf{x} \in \mathbf{R}^n$, $\mathbf{y} \in \mathbf{R}^m$ and $\mathbf{z} = (\mathbf{x},\mathbf{y}) \in \mathbf{R}^{n+m}$. The <u>partial derivative</u> of $\mathbf{F}$ with respect to $\mathbf{y}$ is an m x m block of $\mathbf{F}'(\mathbf{z})$

$$\mathbf{F}_y(\mathbf{z}) = \begin{bmatrix} \partial F_1/\partial y_1 & \partial F_1/\partial y_2 & \cdots & \partial F_1/\partial y_m \\ \vdots & & & \vdots \\ \partial F_n/\partial y_1 & \partial F_n/\partial y_2 & \cdots & \partial F_n/\partial y_m \end{bmatrix}, \quad \text{and is}$$

sometimes denoted by $\partial\mathbf{F}/\partial y_1 \ldots \partial y_m$ or $\partial(F_1,\ldots,F_m)/\partial(y_1,\ldots,y_m)$.

Theorem 6.3 (Implicit Function Theorem): Let $\mathbf{F}:\mathbf{R}^{n+m} \to \mathbf{R}^m$ be a continuously differentiable function. Assume that for some $\mathbf{x}_0 \in \mathbf{R}^n$ and $\mathbf{y}_0 \in \mathbf{R}^m$, $\mathbf{F}(\mathbf{x}_0,\mathbf{y}_0) = 0$ and $\det \mathbf{F}_y(\mathbf{x}_0,\mathbf{y}_0) \neq 0$. Then there is a continuous function $\mathbf{f}:\mathbf{R}^n \to \mathbf{R}^m$ defined on some neighborhood N of $\mathbf{x}_0$ such that $\mathbf{f}(\mathbf{x}_0) = \mathbf{y}_0$ and $\mathbf{F}(\mathbf{x},\mathbf{f}(\mathbf{x})) = \mathbf{0}$ for all $\mathbf{x} \in N$.

Example: Let $\mathbf{F}:\mathbf{R}^5 \to \mathbf{R}^2$, $\mathbf{F} = (f_1,f_2)$ where

$$f_1(x,y,z,u,v) = xy^2 + xzu + yv^2 - 3 \quad \text{and}$$

$$f_2(x,y,z,u,v) = u^3yz - 2xv + u^2v^2 .$$

The derivative is:

$$\mathbf{F}'(x,y,z,u,v) = \begin{bmatrix} y^2+zu & 2xy+v^2 & xu & xz & 2yv \\ -2v & zu^3 & yu^3 & 3u^2yz+2uv^2 & -2x+2u^2v \end{bmatrix} .$$

$\mathbf{F}(1,1,1,1,1) = (0,0)$, so consider the Fréchet derivative there:

$$\mathbf{F}'(1,1,1,1,1) = \begin{bmatrix} 2 & 3 & 1 & 1 & 2 \\ -2 & 1 & 1 & 5 & 0 \end{bmatrix} .$$

At $(1,1,1,1,1)$, $\partial\mathbf{F}/\partial(x,y) = \begin{bmatrix} 2 & 3 \\ -2 & 1 \end{bmatrix}$ and $\det(\partial\mathbf{F}/\partial(x,y)) = 8$.

Therefore, there is some function $\mathbf{f}:\mathbf{R}^3 \to \mathbf{R}^2$ such that $(x,y) = \mathbf{f}(z,u,v)$, and in some neighborhood $\mathbf{N}$ in $\mathbf{R}^3$ of $(z,u,v) = (1,1,1)$ $\mathbf{F}(\mathbf{f}(z,u,v),z,u,v) \equiv 0$. Since $\det(\partial\mathbf{F}/\partial(y,u)) = 15$, there is also some function $\mathbf{g}:\mathbf{R}^3 \to \mathbf{R}^2$ such that $(y,u) = \mathbf{g}(x,z,v)$.

Example: Let $\mathbf{F}:\mathbf{R}^5 \to \mathbf{R}^3$, $\mathbf{F} = (f_1,f_2,f_3)$ where

$f_1(x,y,z,u,v) = 2x + y + 2z + u - v - 1$,

$f_2(x,y,z,u,v) = xy + z - u + 2v - 1$, and

$f_3(x,y,z,u,v) = yz + xz + u^2 + v$.

$\mathbf{F}(1,1,-1,1,1) = \mathbf{0}$ and

$$\mathbf{F}'(1,1,-1,1,1) = \begin{bmatrix} 2 & 1 & 2 & 2 & -1 \\ 1 & 1 & 1 & -1 & 2 \\ -1 & -1 & 2 & 2 & 1 \end{bmatrix} .$$

At $(1,1,-1,1,1)$, $\partial\mathbf{F}/\partial(x,y,z) = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \\ -1 & -1 & 2 \end{bmatrix}$, so $|\partial\mathbf{F}/\partial(x,y,z)| = 3$.

By the implicit function theorem, there is a function $\mathbf{f}:\mathbf{R}^2 \to \mathbf{R}^3$ defined on a neighborhood N of $(1,1)$ such that $(x,y,z) = \mathbf{f}(u,v)$ and $\mathbf{F}(\mathbf{f}(u,v),u,v) \equiv 0$ on N.

Note that in these examples, the implicit function theorem was used to establish the existence of a function, but no attempt was made to find the function explicitly. It is often impossible to do so.

Also note that, like the inverse function theorem, this is a <u>local</u> theorem which only establishes the existence of a function in some region about a particular point.

Theorem 6.4: Let $f: \mathbf{R}^n \to \mathbf{R}^1$, $n > 1$. If f is continuously differentiable, then f is not one-to-one.

Proof: The approach here is to assume that such a function has been found, and to show by contradiction that it cannot exist. If $\nabla f = 0$ everywhere, f is constant and not one-to-one, so there must be some point $\mathbf{x}_0 = (x_1, x_2, \ldots, x_n)$ for which $\nabla f(\mathbf{x}_0) \neq 0$. At this point, the implicit function theorem implies that one coordinate, say x_1, can be written as a function of the other coordinates. Thus $x_1 = g(x_2, \ldots, x_n)$ where g is continuous in some open neighborhood Γ_0 of $\mathbf{x}_0$. Furthermore, $f(g(x_2, \ldots, x_n), x_2, \ldots, x_n) \equiv 0$ on Γ_0. However, if this so then $\nabla f = 0$ on Γ_0 (and at $\mathbf{x}_0$ in particular). Since this contradicts the fact that $\nabla f(\mathbf{x}_0) \neq 0$, it follows that f cannot be continuous and one-to-one.

The inverse and implicit function theorems are fundamental tools in the analysis of higher dimensional dynamical systems, either iterated or continuous.

At this point it is necessary to develop an understanding of some of the basic principles of ordinary differential equations. Most of the features of iterated maps, such as attracting and repelling orbits and hyperbolicity, are analogous to properties of systems of ordinary differential equations in phase space.

The linear system of equations $\mathbf{x}' = \mathbf{A}\mathbf{x}$ can be easily solved by a transformation of coordinates which diagonalizes the matrix A, provided that A is a diagonalizable matrix. (See Problem #3 in Appendix A). In two dimensions the system can be written as

$$x' = ax + by$$
$$y' = cx + dy\ .$$

Now consider the general nonlinear two-dimensional system

$$x' = f(x,y)$$
$$y' = g(x,y).$$

This may be approximated in a small neighborhood about a point (x_0, y_0) by a linear system if $a = \partial f/\partial x$, $b = \partial f/\partial y$, $c = \partial g/\partial x$, and $d = \partial g/\partial y$, where all partial derivatives are evaluated at (x_0, y_0). This is a first order series approximation.

Definition 6.5: Let $\mathbf{F}:\mathbf{R}^n \to \mathbf{R}^n$ define the dynamical system $\mathbf{x}'=\mathbf{F}(\mathbf{x})$. The points where $\mathbf{F}(\mathbf{x}) = \mathbf{0}$ are called <u>critical points</u>, <u>equilibrium points</u>, <u>stationary points</u>, or <u>fixed points</u>.

Theorem 6.5: Let the n-dimensional dynamical system $\mathbf{F}:\mathbf{R}^n \to \mathbf{R}^n$ be defined by $x_i' = f_i(\mathbf{x})$ where $\mathbf{x} \in \mathbf{R}^n$ (so $1 \le i \le n$) and $\mathbf{F}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_n(\mathbf{x}))$. In a neighborhood of an equilibrium point $\mathbf{x}_0$ in the domain of $\mathbf{F}$, the behavior of the system is approximated by the linear system $\mathbf{x}' = \mathbf{F}'(\mathbf{x}_0)\mathbf{x}$ where $\mathbf{F}'(\mathbf{x}_0)$ is the Fréchet derivative of $\mathbf{F}$ at the point $\mathbf{x}_0$.

Obviously, critical points are analogous to the fixed points defined for iterated maps. The behavior of a map near such points can be examined by finding the linearized system as described in Theorem 6.5 and finding the eigenvalues of **F'**. Since the matrix **F'** is different at each critical point, the behavior of each one must be considered separately.

Example: The Lotka-Volterra competition equations are used to model two biological species which interact. They are

$$x' = \beta_1 x(K_1 - x - \alpha_1 y)$$

$$y' = \beta_2 y(K_2 - y - \alpha_2 x)$$

where x and y are the two populations. There are three critical points which are immediately obvious; $(0,0)$, $(0,K_2)$, and $(K_1,0)$. A fourth equilibrium point can found by solving the two equations

$$K_1 = x + \alpha_1 y$$

$$K_2 = y + \alpha_2 x \ .$$

The solution is (x_0,y_0) where $x_o = (K_1 - \alpha_1 K_2)/(1 - \alpha_1\alpha_2)$ and $y_o = (K_2 - \alpha_2 K_1)/(1 - \alpha_1\alpha_2)$. The 'linearization' of the system is $(x,y)^{T\prime} = A(x,y)^T$ where A is the matrix

$$\begin{bmatrix} \beta_1(K_1 - 2x - \alpha_1 y) & -\beta_1\alpha_1 x \\ -\beta_2\alpha_2 y & \beta_2(K_2 - 2y - \alpha_2 x) \end{bmatrix} .$$

At the first three critical points:

$$A(0,0) = \begin{bmatrix} \beta_1 K_1 & 0 \\ 0 & \beta_2 K_2 \end{bmatrix} ,$$

$$A(0,K_2) = \begin{bmatrix} \beta_1(K_1 - \alpha_1 K_2) & 0 \\ -\beta_2\alpha_2 K_2 & -\beta_2 K_2 \end{bmatrix} , \text{ and}$$

$$A(K_1,0) = \begin{bmatrix} -\beta_1 K_1 & -\beta_1\alpha_1 K_1 \\ 0 & \beta_2(K_2 - \alpha_2 K_1) \end{bmatrix} .$$

Since populations are inherently positive, (x_0,y_0) is only of interest if it is in the first quadrant. Since (x_0,y_0) is a

Since populations are inherently positive, (x_0,y_0) is only of interest if it is in the first quadrant. Since (x_0,y_0) is a critical point of the original equations and $x_0 \neq 0$ and $y_0 \neq 0$, $(K_1 - x - \alpha_1 y)$ and $(K_2 - y - \alpha_2 x)$ are both zero so

$$A(x_0,y_0) = \begin{bmatrix} -\beta_1 x_0 & -\beta_1\alpha_1 x_0 \\ -\beta_2\alpha_2 y_0 & -\beta_2 y_0 \end{bmatrix} .$$

Definition 6.6: Let $\mathbf{F}:\mathbf{R}^n \to \mathbf{R}^n$ and let $\mathbf{x}_0$ be a critical point of $\mathbf{F}$. The point $\mathbf{x}_0$ is said to be <u>stable</u> if for every $\epsilon > 0$ there is a $\delta > 0$ such that every solution $\mathbf{x}(t)$ which, for some time t_0, satisfies $\|\mathbf{x}(t_0) - \mathbf{x}_0\| < \delta$, $\|\mathbf{x}(t) - \mathbf{x}_0\| < \epsilon$ for all $t \geq t_0$. If the point $\mathbf{x}_0$ is stable and there is some $\eta > 0$ such that any solution which satisfies $\|\mathbf{x}(t_0) - \mathbf{x}_0\| < \eta$ satisfies $\lim_{t\to\infty}\|\mathbf{x}(t) - \mathbf{x}_0\| = 0$, then $\mathbf{x}_0$ is called an <u>asymptotically stable</u> point. If $\mathbf{x}_0$ is not stable, it is called <u>unstable</u>.

In non-rigorous terms, a critical point is stable if trajectories which are close to the critical point at some time t_0 stay close after that, asymptotically stable if nearby trajectories are attracted to the critical point, and unstable if at least one trajectory which is nearby at t_0 goes away from the point at a later time.

Definition 6.7: Let $\mathbf{F}:\mathbf{R}^n \to \mathbf{R}^n$, so that $\mathbf{x}' = \mathbf{F}'(\mathbf{x}_0)\mathbf{x}$ is the corresponding linearization near a critical point $\mathbf{x}_0$. If the real parts of all of the eigenvalues of $\mathbf{F}'(\mathbf{x}_0)$ are non-zero, then the point $\mathbf{x}_0$ is called a <u>hyperbolic point</u>. Otherwise, it is <u>non-hyperbolic</u>.

Theorem 6.6: If $\mathbf{x}_0$ is a hyperbolic critical point of the system $\mathbf{x}' = \mathbf{F}(\mathbf{x})$, $\mathbf{F}:\mathbf{R}^n \to \mathbf{R}^n$, then the linearized system $\mathbf{x}' = \mathbf{F}'(\mathbf{x}_0)\mathbf{x}$ behaves in a manner qualitatively similar to the original system in some neighborhood of $\mathbf{x}_0$.

Consider the two-dimensional case, so the linearized system has the form $\mathbf{x}' = A\mathbf{x}$ where A is a 2x2 matrix. If the eigenvalues of A, λ_1 and λ_2, are real and distinct a transformation of coordinates T can be found such that $T^{-1}AT$ is diagonal. This is the <u>canonical form</u> of A. If λ_1, λ_2 are not distinct, are complex with non-zero real part, or are pure imaginary a transformation T can be found such that $T^{-1}AT$ is a matrix of appropriate canonical form. Eigenvalues are not changed by these transformations, so only these basic canonical forms need be considered.

<u>Case #1:</u> Real eigenvalues of the same sign, $\lambda_1\lambda_2 > 0$.
For distinct values, $\lambda_1 \neq \lambda_2$, the canonical form is diagonal. Thus the system can be transformed into one where $x' = \lambda_1 x$, $y' = \lambda_2 y$. The solution is $x = x_0 e^{\lambda 1 t}$, $y = y_0 e^{\lambda 2 t}$. If $\lambda_1, \lambda_2 > 0$, then all trajectories are repelled and the critical point is an <u>unstable node</u>. If $\lambda_1, \lambda_2 < 0$, then trajectories are all attracted and the point is a <u>stable node</u>. See Figure 6.1.

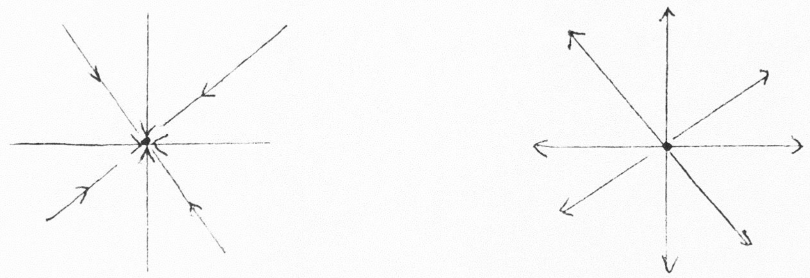


Figure 6.1 - Stable and Unstable Nodes

If $\lambda_1 = \lambda_2$, the canonical form is either $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$ or $\begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix}$.

The conclusion is similar. Obviously, stable nodes are asymptotically stable points and unstable nodes are unstable points.

Case #2: Real eigenvalues of opposite sign, $\lambda_1\lambda_2 < 0$.

Since $\lambda_1 \neq \lambda_2$ the canonical form is always diagonal, and $x = x_0e^{\lambda 1t}$, $y = y_0e^{\lambda 2t}$. Thus the point attracts in one direction and repels in the other. This type of unstable critical point is called a saddle. See Figure 6.2.

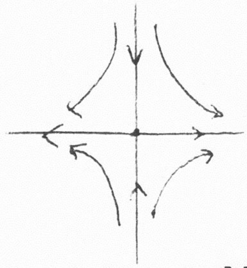


Figure 6.2 - Saddle Point

Case #3: Complex eigenvalues with non-zero real part. ($\lambda_{1,2} = \alpha \pm i\beta$).

The canonical form in this case is $A = \begin{matrix} \alpha & \beta \\ -\beta & \alpha \end{matrix}$, and the solutions are $x,y = e^{\alpha t}(a \cos \beta t \pm b \sin \beta t)$. Thus the real part, α, indicates whether trajectories are attracted or repelled while the imaginary part, β, results in an oscillatory component in the solution. This type of critical point is called a spiral, focal point, or vortex point. If $\alpha < 0$ the point is asymptotically stable, and if $\alpha > 0$ it is unstable. See Figure 6.3.

Case #4: Purely imaginary eigenvalues. (Nonhyperbolic).
This is like the last case with $\alpha = 0$. The point is called a center. It is stable but not asymptotically stable. See Figure 6.3.

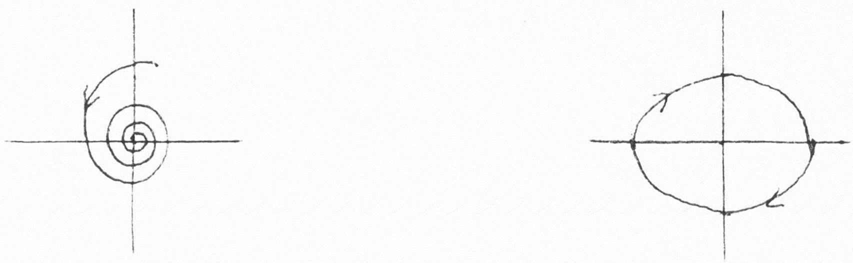


Figure 6.3 - A Stable Spiral and a Center

Two things should be remembered here. First, only the local behavior near the critical point has been determined in the case of a general nonlinear system. Filling in the rest of the phase plane is, in general, a difficult problem. Second, a matrix which is not in canonical form will have features which are distortions of those shown above. While a similarity transformation ($T^{-1}AT$) leaves eigenvalues unchanged, eigenvectors are not invariant. See Figure 6.4.

Definition 6.8: Let $\mathbf{F}:\mathbf{R}^n \rightarrow \mathbf{R}^n$, and let $\mathbf{x}_0$ be a critical point of $\mathbf{F}$. The index of $\mathbf{x}_0$ is the number of eigenvalues of $\mathbf{F}'(\mathbf{x}_0)$ with positive real part.

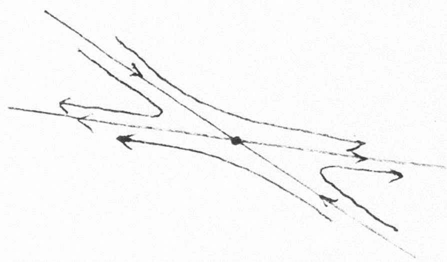


Figure 6.4 - General Saddle Point

For an n-dimensional system, a critical point with index n is repelling (in all directions). A non-hyperbolic critical point with index 0 is attracting (in all directions) and is asymptotically stable. If the index is not 0 or n, a non-hyerbolic point attracts in some directions and repels in others. In this case it is not stable, since the linear system repels on some subspace $\mathbf{R}^m$, $m < n$. For $n = 3$, see Figure 6.5.

Nonlinear systems also have limit cycles, which are periodic solutions (closed trajectories) which attract or repel nearby trajectories. In $\mathbf{R}^2$, the Poincaré-Bendixson Theorem and related results assist in the analysis of limit cycles. For higher dimensions, however, there are no analogous results. For an example of a system with a (countably) infinite number of distinct limit cycles, see problem #7 at the end of the chapter.

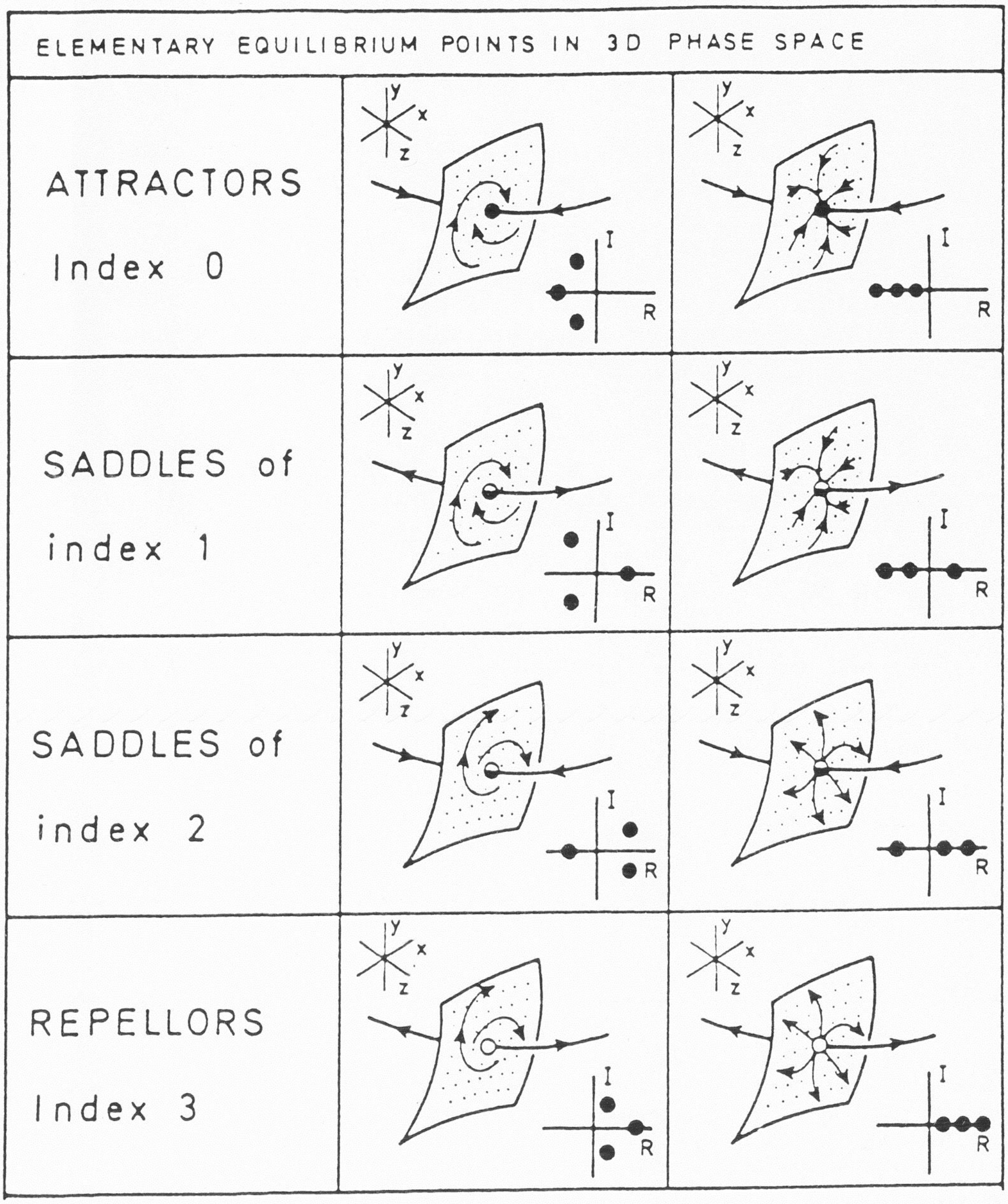


Figure 6.5 - Critical points in $\mathbf{R}^3$

Further Reading

The inverse and implicit function theorems are covered in detail in most books on advanced calculus, including [Apostol]. The introduction to differential equations presented here is, of necessity, brief and cursory. [Waltman] covers most of the topics in this chapter in more detail and is quite readable, requiring only the standard introductory courses in linear algebra and differential equations.

4) Let T define a change of coordinates, $\mathbf{y} = T\mathbf{x}$. What condition is necessary on T for this to be a homeomorphism?

5) For the real-valued matrices of interest here, complex eigenvalues always occur in conjugate pairs. Why?

6) Show that the characteristic polynomial of A is invariant under a similarity transformation, $B = T^{-1}AT$. [Hint: Start with $p_B = \det(\lambda I - B)$ and then let $I = T^{-1}T$ and $B = T^{-1}AT$.]

7) Consider the system:

$$x' = -y + x(x^2 + y^2)\sin(\pi/(x^2 + y^2)^{1/2})$$

$$y' = x + y(x^2 + y^2)\sin(\pi/(x^2 + y^2)^{1/2}) .$$

a) Convert the system to polar coordinates and show that $r = 1/n$, $\theta = t$ is a solution.

b) Show that $r' > 0$ for $1/(2n + 1) < r < 1/2n$ and $r' < 0$ for $1/2n < r < 1/(2n-1)$.

c) Make a sketch in the phase plane for the first few values of n.

8) Analyze the stability of the critical points in the Lotka-Volterra equations.

Problems

1) Let $f(x) = (x/2) + x^2\sin(1/x)$.

a) Explain why $f(x)$ is not invertible in the neighborhood $(-\epsilon,\epsilon)$ for any ϵ. [Hint: Is it one-to-one?]

b) Show that $f'(0) \neq 0$. (Use the definition; $f'(x) = \lim_{h\to 0} h^{-1}(f(x+h) - f(x))$.) Why does this not contradict the implicit function theorem?

2) Let $f(x,y) = xe^y - y + 1$. Determine whether or not the curve $f(x,y) = 0$ can be represented in the forms $x = f(y)$ and $y = g(x)$ near the point $(e^{-2},2)$.

3) Let $F(\mu,x) = \mu x(1 - x)$.

a) Find the Fréchet derivative F', and the partial derivative submatrices of F' for $\partial F/\partial\mu$ and $\partial F/\partial x$.

b) Where does the implicit function theorem allow us to write $\mu = f(x)$? $x = g(\mu)$?

c) At the points where the implicit function theorem does not allow a conclusion, show that the functions f or g cannot exist. [Hint: Implicitly differentiate to find $\mu'(x)$ and $x'(\mu)$ and draw your conclusions from these. Find μ'' and x'' if necessary.]

4) Let T define a change of coordinates, $\mathbf{y} = T\mathbf{x}$. What condition is necessary on T for this to be a homeomorphism?

Chapter 7 - Introduction to Continuous Systems

In this chapter some of the basic features of continuous systems are introduced. In the last chapter, the behavior a system near a critical point was shown to depend on the eigenvalues of the linearized system at the point. These were divided into three types; those with positive real part, those with negative real part, and those with zero real part. In the direction of the corresponding eigenvectors, the point was seen to be unstable, asymptotically stable, and indeterminate, respectively.

Let $A:\mathbf{R}^n \to \mathbf{R}^n$ be a linear transformation, and consider $\mathbf{x}' = A\mathbf{x}$. Then there will be n eigenvalues and n (generalized) eigenvectors associated with A (counting multiplicities). A set of m eigenvectors will span a subspace, $\mathbf{R}^m$, of $\mathbf{R}^n$. The space $\mathbf{R}^n$ can be partitioned into three subspaces in the following way.

Definition 7.1: Let $A:\mathbf{R}^n \to \mathbf{R}^n$ be a linear transformation, so $\mathbf{x}' = A\mathbf{x}$. Let v^i denote an eigenvector corresponding to an eigenvalue with positive real part, u^i denote an eigenvector corresponding to eigenvalue with negative real part, and w^i denote an eigenvector corresponding to eigenvalue with zero real part. Then $E^s = \text{span}\{v^1,\ldots,v^{ns}\}$, $E^u = \text{span}\{u^1,\ldots,u^{nu}\}$, and $E^c = \text{span}\{w^1,\ldots,w^{nc}\}$ where $n_s + n_u + n_c = n$. E^s is called the <u>stable subspace</u> of A, E^u is called the <u>unstable subspace</u> of A, and E^c is called the <u>center subspace</u> of A.

For a general nonlinear system there is generally more than one fixed point and the space $\mathbf{R}^n$ cannot be divided up so nicely into subspaces. Instead, there are manifolds associated with each distinct fixed point. For the remainder of this chapter, critical points will be assumed to be hyperbolic. Thus the center subspace and its corresponding manifold will not be of interest here. They turn out to be very important in the study of bifurcations.

Definition 7.2: Let $\mathbf{F}:\mathbf{R}^n \to \mathbf{R}^n$, $\mathbf{x}' = \mathbf{F}(\mathbf{x})$. Let $\mathbf{x}_0$ be a critical point of $\mathbf{F}$ and N be a neighborhood of $\mathbf{x}_0$. The local stable manifold of $\mathbf{x}_0$, $W^s_{loc}(\mathbf{x}_0)$, and the local unstable manifold of $\mathbf{x}_0$, $W^u_{loc}(\mathbf{x}_0)$, are defined by

$$W^s_{loc}(\mathbf{x}_0) = \{ \mathbf{x} = \mathbf{x}(t_0) \in N \mid \lim_{t\to\infty} \|\mathbf{x}(t) - \mathbf{x}_0\| = 0 \}$$

and

$$W^u_{loc}(\mathbf{x}_0) = \{ \mathbf{x} = \mathbf{x}(t_0) \in N \mid \lim_{t\to-\infty} \|\mathbf{x}(t) - \mathbf{x}_0\| = 0 \} .$$

These manifolds are closely related to the subspaces of Definition 7.1 at the critical point.

Theorem 7.1 (Stable Manifold Theorem): Let $\mathbf{F}:\mathbf{R}^n \to \mathbf{R}^n$ and let $\mathbf{x}_0$ be a hyperbolic fixed point of $\mathbf{F}$. Then there are local stable and unstable manifolds $W^s_{loc}(\mathbf{x}_0)$, $W^u_{loc}(\mathbf{x}_0)$ which have the same dimensions n_s, n_u as those of the eigenspaces E^s, E^u of the linearized system $\mathbf{F}'(\mathbf{x}_0)$. Furthermore, $W^s_{loc}(\mathbf{x}_0)$ and $W^u_{loc}(\mathbf{x}_0)$ are tangent to E^s and E^u, respectively, at $\mathbf{x}_0$.

The global stable and unstable manifolds can be found by following points in W_{loc} forward (backwards) in time. (Forward for W^u, backwards for W^s.)

Example: Let $x' = x$, $y' = -y + x^2$. This system has a single critical point at $\mathbf{x}_0 = (0,0)$. The linearized system has the diagonal matrix $(1,-1)$ as $\mathbf{F}'(\mathbf{x}_0)$. For $\lambda_1 = 1$, $u^1 = (1,0)$ and for $\lambda_2 = -1$ $v^1 = (0,1)$. Thus E^s is the y axis and E^u is the x axis. The global manifolds can be found by integration in this case. Dividing y' by x' gives the <u>phase plane equation</u>

$$dy/dx = -(y/x) + x.$$

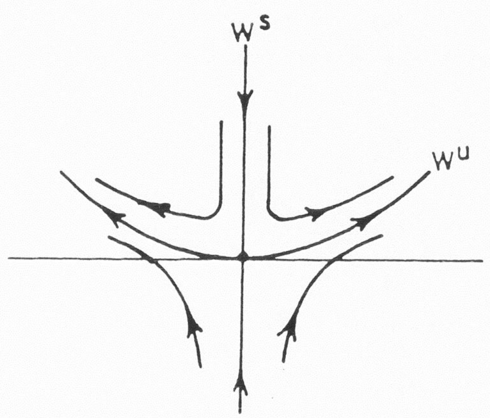
Rewriting this as

$$xy' + y = x^2 , \quad \text{or as} \quad (xy)' = x^2,$$

it can be directly integrated. The solution is

$$xy = x^3/3 + c, \quad \text{or} \quad y = x^2/3 + c/x .$$

The trajectory through $\mathbf{x}_0$ tangent to E^u is $y = x^2/3$ and the trajectory tangent to E^s is $x = 0$. See Figure 7.1.



<u>Figure 7.1 - Example</u>

Limit cycles can also be studied for stability. One way of doing this is the <u>Poincaré map</u>, also called a <u>first return map</u>. This important technique will only be introduced briefly here.

Let γ denote a periodic trajectory. Choose a point $\mathbf{p}$ on γ. Now take a local cross section Σ of dimension n-1 such that Σ contains $\mathbf{p}$ and $\mathbf{F}(\mathbf{x})\cdot\mathbf{n}(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in \Sigma$, where $\mathbf{n}(\mathbf{x})$ is the unit normal vector to Σ at $\mathbf{x}$. (This last condition prevents Σ from being tangent to any trajectories.) Let $U \subset \Sigma$ denote some neighborhood of $\mathbf{p}$. The Poincaré map $P:U \rightarrow \Sigma$ is given by $P(\mathbf{q}) = \mathbf{q}_\tau$ where $\mathbf{q}_\tau$ indicates the point on Σ where the trajectory through $\mathbf{q}$ first returns (after time τ). The time interval τ will depend on $\mathbf{q}$, but will approach the period of γ as $\mathbf{q} \rightarrow \mathbf{p}$. If the orbit of $\mathbf{q}$ in Σ approaches $\mathbf{p}$ for all $\mathbf{q}$ in some neighborhood U of $\mathbf{p}$, then γ is asymptotically stable. The iterated map $P(\mathbf{q})$ can be analyzed using the methods already developed.

Poincaré maps can be computed if the general solution in an appropriate region is known. They can also be approximated using perturbation and averaging methods.

Certain sets of points in phase space have properties which make them important and/or interesting.

Definition 7.3: Let $\mathbf{F}:\mathbf{R}^n \rightarrow \mathbf{R}^n$. Let $\mathbf{x}(t)$ be a trajectory of $\mathbf{F}$ in the phase plane with $\mathbf{x}(t_0) = \mathbf{x}_0$. A set S of points $\mathbf{x}_0$ such that $\mathbf{x}(t) \in S$ for all $t \geq t_0$ is called an <u>invariant set</u> of $\mathbf{F}$. Let U be a neighborhood of $\mathbf{x}_0 = \mathbf{x}(t_0)$. If, for any U, there is a point $\mathbf{y}_0 = \mathbf{y}(t_0) \in U$ such that $\mathbf{y}(t) \in U$ for some $t > t_0$, then $\mathbf{x}_0$ is called a <u>nonwandering point</u> of $\mathbf{F}$. The set of all such points is called the <u>nonwandering set</u> of $\mathbf{F}$ and is denoted by Ω.

A point is in an invariant set S if its trajectory is contained in S. The global stable and unstable manifolds are invariant sets. A nonwandering point either returns to within some small distance of itself, or is a limit point of such points. Fixed points and periodic orbits are contained in Ω.

Example: Consider the undamped linear oscillator, $x'' + x = 0$. Every point lies on a periodic orbit, so $\Omega = \mathbf{R}^2$. Since all trajectories are closed, each is a bounded invariant set. The union of all invariant sets is $\mathbf{R}^2$. This is best visualized by considering the phase space to be the surface of a cylinder. See Figure 7.2, and imagine it is on the surface of a cylinder so that the left and right sides are attached.

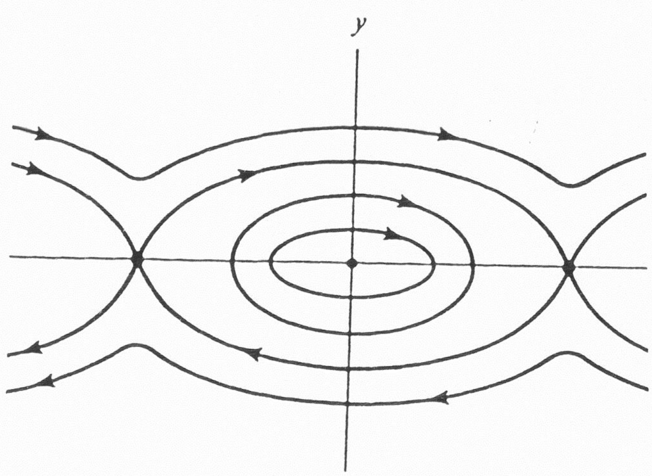


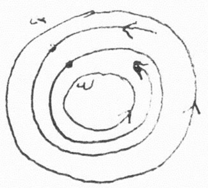
Figure 7.2 - Linear Oscillator

Definition 7.4: Let $\mathbf{F}:\mathbf{R}^n \to \mathbf{R}^n$ and let $\mathbf{x}_0 = \mathbf{x}(t_0)$ be a point in $\mathbf{R}^n$. A point $\mathbf{p}$ is an <u>ω-limit point</u> of $\mathbf{x}_0$ if there is a sequence of points $\mathbf{x}(t_1),\mathbf{x}(t_2),\ldots$ on the trajectory of $\mathbf{x}_0$ such that $\mathbf{x}(t_i) \to \mathbf{p}$

and $t_i \to \infty$. A point $\mathbf{q}$ is an <u>α-limit point</u> of $\mathbf{x}_0$ if $\mathbf{x}(t_i) \to \mathbf{q}$ and $t_i \to -\infty$. The set of all ω-limit (α-limit) points of $\mathbf{x}_0$ is called the <u>ω-limit set</u> (<u>α-limit set</u>) of $\mathbf{x}_0$.

Example: Consider a two dimensional critical point $\mathbf{x}_0$ which is a saddle, and assume it has unique (non-intersecting) global stable and unstable manifolds. Then $\mathbf{x}_0$ is an ω-limit point of all $\mathbf{x} \in W^s$, and is an α-limit point of all $\mathbf{x} \in W^u$.

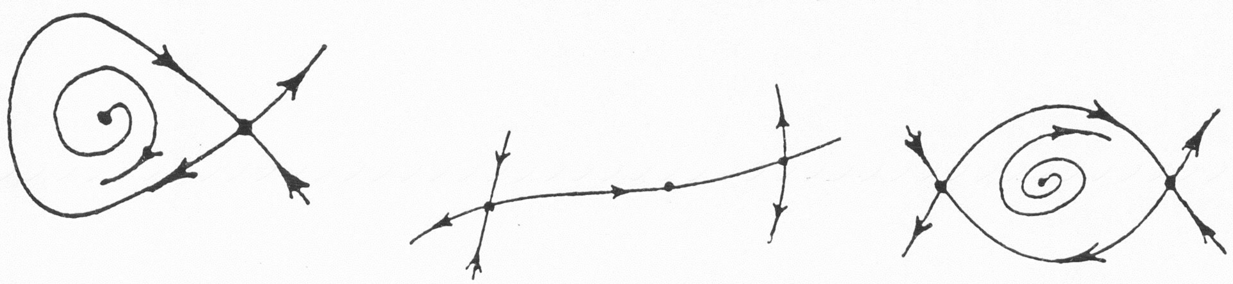
Example: Consider a system in two dimensions with concentric periodic trajectories. (See Figure 7.3). Assume that trajectories between the two limit cycles tend away from the outer one and towards the inner one. Then for all points in this annular region, the outer cycle is the α-limit set and the inner cycle is the ω-limit set. The sequence of points in the definition can be taken to be an orbit on a (one dimensional) Poincaré map.



<u>Figure 7.3 - α,ω-limit sets</u>

Finally, there are several types of trajectories which are particularly important.

Definition 7.5: A trajectory which connects distinct critical points is called heteroclinic. One which connects a critical point to itself is called homoclinic. Closed paths formed of heteroclinic trajectories are called homoclinic cycles. (See Figure 7.4).



(i) Homoclinic (ii) Heteroclinic (iii) Homoclinic Cycle

Figure 7.4 - Special Trajectories

Further Reading

The material in this chapter is covered more thoroughly and with a higher level of mathematical rigor in both [Guckenheimer/Holmes] and [Hirsch/Smale].

Problems

1) Can two stable manifolds of distinct critical points intersect? What about unstable ones? Explain why or why not.

2) Consider a linear two-dimensional system with one critical point at the origin, which is a saddle. Describe the invariant sets S and the nonwandering set Ω. Is any invariant set contained entirely in Ω?

3) Let $F_\mu = \mu x(1 - x)$. How many bounded invariant sets are there? Describe it (them) for $\mu = 4$ and for $\mu > 4$.

4) Describe the α-limit and ω-limit sets for all points in the phase plane for the system given in Chapter 6, Problem 7.

Chapter 8 - Structural Stability

In the last chapter, the stability of critical points was examined in a local sense. Whether or not a critical point is stable depends on what happens to nearby points as time changes. In this chapter a type of global stability, or robustness, is described. A dynamical system is structurally stable if "nearby" systems behave in the same way. Structural instability is often associated with the existence of non-hyperbolic critical points.

Example: Consider the simple undamped oscillator $x'' + \sin x = 0$. Letting $y = x'$, this can be written as

$$x' = y$$
$$y' = -\sin x \; .$$

The critical points are $(n\pi, 0)$. The linearized system is:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\cos n\pi & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} .$$

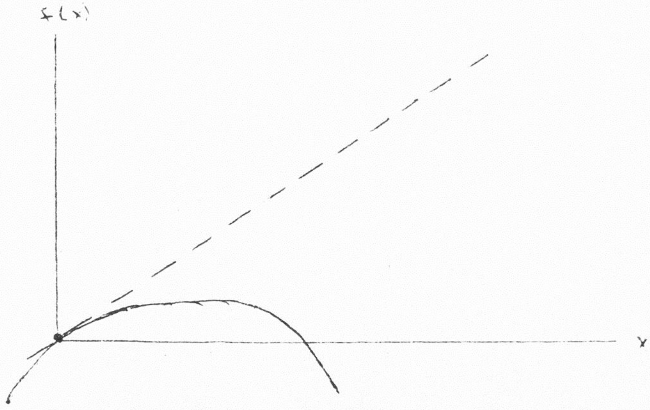
The eigenvalues are $\pm i$ if n is even and ± 1 if n is odd. Thus the points $((2n), 0)$ are non-hyperbolic. This equation is structurally unstable since the addition of any arbitrarily small amount of damping changes the nature of the entire phase portrait.

To put the idea of structural stability in rigorous terms, "nearby" and "same behavior" must be defined. The second one is easy; two systems have the same dynamics if they are topologically conjugate.

Definition 8.1: Let $\mathbf{F}:\mathbf{R}^n \to \mathbf{R}^n$, $\mathbf{F} \in C^r$. The <u>C^k-norm</u> ($k \le r$), denoted by $\|\mathbf{F}\|_k$, is defined as the least upper bound of the set of numbers $|\mathbf{F}(\mathbf{x})|$, $\|\mathbf{F}'(\mathbf{x})\|$, $\|\mathbf{F}''(\mathbf{x})\|$, $\cdots$, $\|\mathbf{F}^{(k)}(\mathbf{x})\|$ over all $\mathbf{x} \in \mathbf{R}^n$. If $\|(\mathbf{F} - \mathbf{G})(\mathbf{x})\|_k < \epsilon$, $\mathbf{G}(\mathbf{x})$ is called a <u>C^k-perturbation of size ϵ</u> of $\mathbf{F}$.

Definition 8.2: A system $\mathbf{F}:\mathbf{R}^n \to \mathbf{R}^n$, $\mathbf{F} \in C^r$, is <u>C^r-structurally stable</u> if there is an $\epsilon > 0$ such that C^r-perturbations of size ϵ are topologically conjugate to $\mathbf{F}$.

Example: Let $f_\epsilon(x) = x - x^2 + \epsilon$, so $f_0 = x(1 - x)$. $\|(f_\epsilon - f_0)(x)\|_k$ = l.u.b. $\{ |\epsilon|, 0 \} = |\epsilon|$, so f_ϵ is a C^k-perturbation of size ϵ for any k. It is easy to see graphically that $f_0(0) = 0$ is a fixed point. Note that it is non-hyperbolic. See Figure 8.1. For $\epsilon > 0$, f_ϵ has two fixed points and for $\epsilon < 0$ it has none. Since the dynamics of f_ϵ differ fundamentally from those of f_0, the two will not be topologically conjugate. Therefore, $f(x) = x(1 - x)$ is not structurally stable.



<u>Figure 8.1 - f(x) = x(1 - x)</u>

There are several useful results which state that certain types of systems are structurally stable.

Definition 8.3: Let $\mathbf{F}:\mathbf{R}^n \to \mathbf{R}^n$. If $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ can be written in the form $\mathbf{x}' = -\text{grad } V(x)$ for some potential function $V:\mathbf{R}^n \to \mathbf{R}^n$, then $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ is called a <u>gradient system</u>.

Example: Consider the system in $\mathbf{R}^2$ given by

$$x' = -ax + xy$$

$$y' = -ay + 1/2(x^2 - y^2) \ .$$

This can be written as $x' = -\partial V/\partial x$, $y' = -\partial V/\partial y$, or $(x,y)' = -\text{grad } V(x,y)$ where $V:\mathbf{R}^2 \to \mathbf{R}$,

$$V(x,y) = (a/2)(x^2 + y^2) + (1/2)(y^3/3 - x^2y).$$

Theorem 8.1 (Palis/Smale): Gradient systems for which all fixed points are hyperbolic and all intersections of stable and unstable manifolds transverse are structurally stable.

The following result holds on all systems on two dimensional manifolds. It is stated here in terms of $\mathbf{R}^2$.

Theorem 8.2 (Peixoto): Let $\mathbf{F}:\mathbf{R}^2 \to \mathbf{R}^2$, $\mathbf{F} \in C^r$. The system $\mathbf{F}$ is structurally stable if and only if the following conditions hold.

1) The number of fixed points and closed trajectories is finite and each is hyperbolic.
2) There are no trajectories connecting saddle points.
3) The nonwandering set consists only of fixed points and periodic trajectories.

Example: The simple oscillator of the first example in this chapter is not structurally stable since there are (heteroclinic) trajectories connecting saddle points. If the system is viewed as being defined on the surface of a cylinder (which is a two dimensional manifold), the number of fixed points is not infinite as it is in the plane. However, the number of closed trajectories is not finite.

Smale attempted to study systems similar to the type above, with a modification of the second condition.

Definition 8.4: A <u>Morse-Smale</u> system is one in which:

1) The number of fixed points and closed trajectories is finite and each is hyperbolic ,
2) All stable and unstable manifolds intersect transversally, and
3) The nonwandering set consists only of fixed points and periodic trajectories.

Theorem 8.3: Morse-Smale systems are structurally stable.

While the original conjecture was that a system is structurally stable if and only if it is Morse-Smale, the converse of Theorem 8.3 is not true. Smales' famous horseshoe map provided a counter-example.

Structural stability is clearly a desirable feature of physical models. Higher order effects are often neglected when models are built. If it turns out that these effects change the basic nature of the way the model behaves, one must question the value of the model. Despite its desirability, many important and interesting models are not structurally stable.

Further Reading

[Devaney] contains a section on structural stability of maps on $\mathbf{R}^1$ which is on a level similar to that of the material in this chapter. [Guckenheimer/Holmes] contains a section in Chapter 1 which deals primarily with systems in $\mathbf{R}^n$ and is considerably more mathematical.

Problems

1) Use the criteria given in Chapter 8 to determine whether or not the following are structurally stable.

a) $x' = y$

$y' = x$

b) $x' = -3y \sin x$

$y' = -1 + y^2 + x^2 e^{\cos 2x}$

c) $x' = x \exp[(x^2 + y^2)]$

$y' = y \exp[(x^2 + y^2)]$

d) Chapter 6, problem 7

2) Find an iterated map T_λ which is not structurally stable for any value of λ.

Chapter 9 - Bifurcations

In this chapter functions of two variables will be discussed, $G(x,\lambda) = f_\lambda(x)$. This is a function of x with one parameter, λ. $F_\mu = \mu x(1 - x)$ and $F_c = x^2 + c$ are examples. As the parameter λ is varied, certain values may be found at which the nature of the solution undergoes major changes, or bifurcates. A change in the periodic point structure is usually involved. In the first part of the chapter one-dimensional iterated maps are used to illustrate some basic bifurcation theory.

In the last chapter it was shown that when a system is structurally unstable, a small change in the system causes it to behave very differently. Thus the idea of structural instability is intimately related to bifurcations.

Definition 9.1: Let $\mathbf{f}_\lambda:\mathbf{R}^n \to \mathbf{R}^n$, $\mathbf{x} \in \mathbf{R}^n$, $\lambda \in \mathbf{R}^k$, and $\mathbf{x'} = \mathbf{f}_\lambda(\mathbf{x})$. A value of λ, say λ_0, for which $\mathbf{f}_\lambda$ is structurally unstable is called a <u>bifurcation value</u> of λ.

Example: For $F_c(x) = x^2 + c$, the parameter c has a bifurcation value at 1/4. For $c > 1/4$, the graph of F_c lies above the diagonal. At $c = 1/4$ it is tangent to the diagonal at $x = 1/2$. When $c < 1/4$, F_c has two fixed points. In the first case all trajectories diverge to $+ \infty$. In the second case, orbits of points in $(-1/2,1/2)$ are attracted to the fixed point and all other orbits (except those of 1/2 and -1/2) diverge. For the third case (as long as $x > -3/4$), there are two fixed points, one repelling and one attracting. The behavior in phase space is shown in Figure 9.1.

(i) $c > 1/4$

(ii) $c = 1/4$

(iii) $c < 1/4$

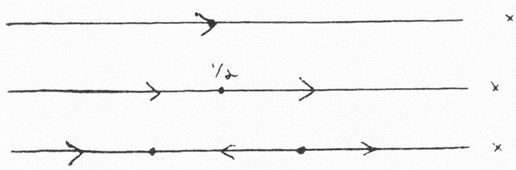


Figure 9.1 - $F_c = x^2 + c$

This type of bifurcation is called a saddle-node or tangent bifurcation. A bifurcation diagram can be drawn by plotting the fixed points on an x vs. c graph. See Figure 9.2.

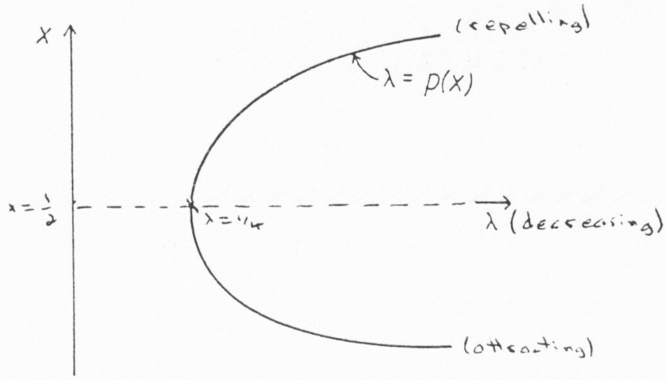
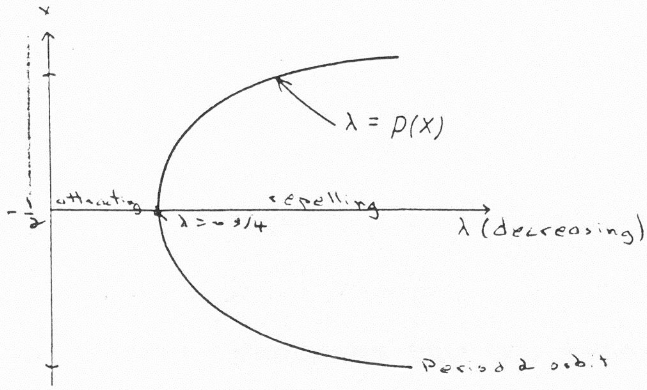


Figure 9.2 - Saddle-Node Bifurcation Diagram, $F_{1/4}(x)$

Example: Again consider $F_c(x)$ and let $c = -3/4$. The two fixed points are $x = -1/2$ and $x = 3/2$. The larger one is repelling, but the smaller one is non-hyperbolic. A bit of analysis shows that for $c > -3/4$ it is attracting, but at the bifurcation point it becomes repelling. However, there is more occurring. For $c = -3/4 - \epsilon$, F_c^2 has two additional fixed points. Thus there is a new periodic orbit with period $T = 2$. This type of bifurcation is called a period-doubling bifurcation. See Figure 9.3.



<u>Figure 9.3 - Period-Doubling Bifurcation</u>

The following theorem states that bifurcations only occur at non-hyperbolic fixed points or periodic points. It holds for periodic orbits since f can be replaced by f^n. It says that in a neighborhood of a hyperbolic fixed point (periodic point), the location of the fixed point will vary continuously with a variation in parameter (λ) and that this fixed point is unique. (Refer to Figures 9.2 and 9.3).

Theorem 9.1: Let f_λ be a one-parameter family of functions and let x_0 be a hyperbolic fixed point of $f_{\lambda 0}$, so $f_{\lambda 0}(x_0) = x_0$ and suppose $|f_{\lambda 0}'(x_0)| \neq 1$. Then for some interval I such that $x_0 \in I$ and some interval N such that $\lambda_0 \in N$ there is a smooth function $p: N \to I$ such that $x_0 = p(\lambda_0)$ and $f_\lambda(p(\lambda)) = p(\lambda)$. In addition, f_λ has no other fixed points in I.

Proof: Let $G(x,\lambda) = f_\lambda(x) - x$. Since x_0 is a fixed point of $f_{\lambda 0}$, $G(x_0,\lambda_0) = 0$. Also, $\partial G/\partial x(x_0,\lambda_0) = f_{\lambda 0}'(x_0) - 1 \neq 0$. By the implicit function theorem, there are intervals I about x_0 and N about λ_0 and a smooth function p such that $p(\lambda_0) = x_0$ and $G(p(\lambda),\lambda) = 0$ for $\lambda \in$ N. $G(x,\lambda) \neq 0$ unless $x = p(\lambda)$. ■

The following two theorems describe the two types of bifurcations introduced earlier. They can be proven by applying the implicit function theorem and the chain rule.

Theorem 9.2 (Saddle-Node Bifurcation): Suppose the following are true:

1) $f_{\lambda 0}(x_0) = x_0$
2) $f_{\lambda 0}'(x_0) = 1$
3) $f_{\lambda 0}''(x_0) \neq 0$
4) $\partial f_\lambda / \partial \lambda |_{\lambda=\lambda 0} \neq 0$.

Then there exists an interval I about x_0 and a smooth function $\lambda=p(x)$, $p:I \rightarrow \mathbf{R}$, such that $f_{p(x)}(x) = x$. Also, $p' = 0$ and $p'' \neq 0$. (See Figure 9.2).

Theorem 9.3 (Period-Doubling Bifurcation): Suppose the following are true:

1) $f_\lambda(x_0) = x_0$ for all λ in a neighborhood of λ_0
2) $f_{\lambda 0}'(x_0) = -1$
3) $f_{\lambda 0}'''(x_0) \neq 0$
4) $\partial f_\lambda^2/\partial\lambda|_{\lambda=\lambda 0} \neq 0$.

Then there exists an interval I about x_0 and a smooth function $p:I \to \mathbf{R}$ such that $f_{p(x)}(x) \neq x$ but $f_{p(x)}^2(x) = x$. (See Figure 9.3).

The saddle-node and period-doubling bifurcations are the most common of those occurring on one-dimensional maps. Some maps will undergo a series of period-doubling bifurcations as a parameter is varied, ultimately reaching a point at which the behavior is chaotic. This is termed the 'period-doubling route to chaos'.

In higher dimensions, the following theorem is important in the study of bifurcations. It is stated for continuous systems here.

Theorem 9.4 (Center Manifold Theorem): Let $\mathbf{F}:\mathbf{R}^n \to \mathbf{R}^n$, $\mathbf{F} \in C^r$, $\mathbf{F}(\mathbf{x}_0) = 0$, and $A = \mathbf{F}'(\mathbf{x}_0)$. Divide the eigenvalues of A into three sets, σ_i.[3]

$\sigma_s = \{\lambda \mid Re(\lambda) < 0\}$

$\sigma_u = \{\lambda \mid Re(\lambda) > 0\}$

$\sigma_c = \{\lambda \mid Re(\lambda) = 0\}$.

Denote the eigenspaces associated with these sets by E^s, E^u, and E^c respectively. Then there exist C^r stable and unstable manifolds W^s and W^u which are tangent to E^s and E^u at $\mathbf{x}_0$, and a C^{r-1} center manifold W^c tangent to E^c at $\mathbf{x}_0$. W^u, W^s, and W^c are all invariant for $\mathbf{F}$. W^s and W^u are unique, but W^c may not be.

Example: Consider the following system in $\mathbf{R}^2$: $x' = x^2$, $y' = -y$. The phase plane equation is $dy/dx = -y/x^2$, or $y' = (-1/x^2)y$. The solution to this is $y = ce^{1/x}$. For $x < 0$, trajectories approach the origin 'flatly', with zero derivatives at the origin. The unique stable manifold is the y axis. However, the center manifold, which must be tangent to the x axis at the origin, is obviously not unique. See Figure 9.4.

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[3] The symbol σ is often used to denote the set of all eigenvalues λ_i of a matrix. This set is called the <u>spectrum</u>.

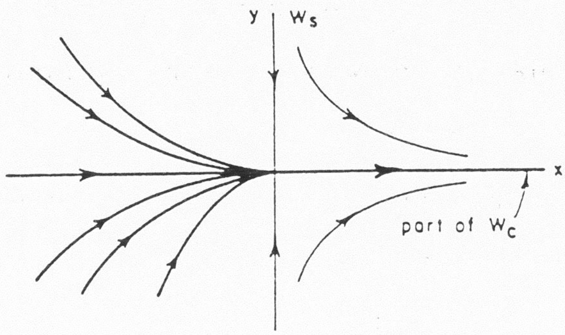


Figure 9.4 - Example

The center manifold theorem is important because it permits many problems of interest related to bifurcation points to be reduced from n-dimensional problems to n_c-dimensional problems, where n_c is the dimension of the center manifold.

Further Reading

Additional materials on bifurcations in iterated maps can be found in [Devaney], including some examples of less typical bifurcations in maps on $\mathbf{R}^1$ and a chapter on the Hopf bifurcation. [Guckenheimer/Holmes] treats bifurcations in n-dimensional dynamical systems.

Problems

1) Follow these steps to prove Theorem 9.2.

a) Show the existence of p such that $f_{p(x)}(x) = x$ by an argument similar to that used in the proof of Theorem 9.1. (Let $G(x,\lambda) = f_\lambda(x) - x$.)

b) Take the derivative of $G(x,\lambda)$ with respect to x by using the chain rule. Solve for $p'(x)$ and show that $p'(x_0)=0$.

c) Use the quotient rule to find $p''(x_0) \neq 0$.

2) This problem illustrates how a lack of smoothness arises when curves are pieced together, as is sometimes done when forming a center manifold. Let $x' = ax$, $y' = by$, $b > a > 0$.

a) Find the critical point(s) and show that it (they) are hyperbolic.

b) Find the phase plane equation and show that $y = C|x|^{b/a}$ is a solution.

c) Form a curve by taking the union of the origin and two curves to the left and right. How many times differentiable will this curve be?

3) Consider the system $x' = \mu x - x^3$, $y' = y$, $\mu' = 0$.

a) Find the equilibrium points. Show that (0,0,0) is non-hyperbolic, but that all others are hyperbolic in the appropriate μ = constant plane. (Note that all μ = constant planes are invariant.)

b) Analyze the stability of each.

c) Find the center, stable, and unstable manifolds of (0,0,0).

d) Hint: A sketch in the x-y-μ phase space may be helpful, but a bit tricky to sketch.

Chapter 10 - The Smale Horseshoe

The famous horseshoe map of Smale is a structurally stable map with infinitely many periodic points. Thus it serves as a counter-example for the converse of Theorem 8.3. It also builds on the method of using sequence spaces, introduced in chapters 3 and 4.

The horseshoe map takes points in a square into the plane, or **H**:**S** $\rightarrow$ **R**^{2} where **S** = [0,a] X [0,a]. The invariant set of **H** will consist of points in **S** which remain in **S** for all iterates of **H**. **H** takes **S** and stretches its height by a factor of μ and scales its width by λ, where $\mu > 2$ and $\lambda < 1/2$. This rectangle is bent into a horseshoe shape so that the two 'legs' are contained is **S**. Figure 10.1 shows one iterate of **H**.

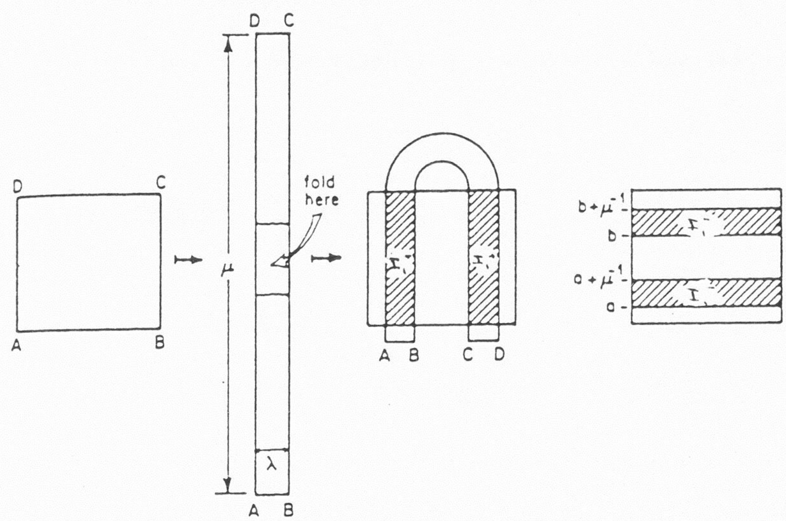


Figure 10.1 - Horseshoe Map

Let I_1^+ indicate the set of points in **H**(**S**)$\cap$**S**. Thus I_1^+ consists of two vertical bars. If the map is "followed backwards", it is seen that $\mathbf{H}^{-1}(I_1^+)$ is made up of two horizontal bars. Call this set I_1^-. These regions are shaded in Figure 10.1.

Let I_2^+ be the set of points in $\mathbf{H}^2(\mathbf{S})\cap\mathbf{S}$. If S with I_1^+ shaded is subject to a second iteration of $\mathbf{H}$, I_2^+ is seen to consist of four vertical bars. Call $\mathbf{H}^{-1}(I_2^+)$ I_2^- so that I_2^- consists of four horizontal bars. See Figure 10.2.

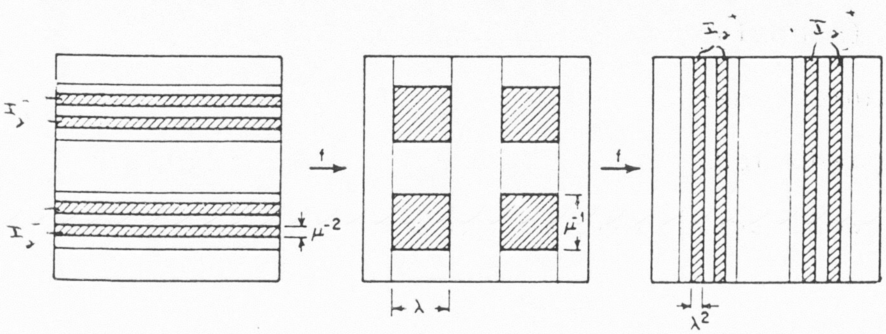


Figure 10.2 - Horseshoe Map, 2nd Iterate

Now let $I^+ = \cap_n I_n^+$ and $I^- = \cap_n I_n^-$. The set I^+ consists of vertical strips which intersect any horizontal line in a Cantor set, and I^- consists of horizontal strips which intersect any vertical line in a Cantor set. The invariant set $I = I^- \cap I^+$ is a two-dimensional Cantor set, Λ_H. Thus $H:\Lambda_H \rightarrow \Lambda_H$, and points not in Λ_H are mapped out of S at some iterate.

This is reminiscent of the maps F_c and F_μ studied earlier, for certain values of c and/or μ. Indeed, this map can also be analyzed through the use of symbolic dynamics, i.e. by using a topologically conjugate sequence space. Let Σ'_2 be the space of all bi-infinite binary sequences, $\cdots a_{-2}a_{-1}.a_0a_1a_2\cdots$. The topological conjugacy, $h(\mathbf{x})$, between Λ_H and Σ'_2 is defined as follows: $h(\mathbf{x}) = (\cdots a_{-2}a_{-1}.a_0a_1\cdots)$ where $a_i = 0$ $\mathbf{H}^i(\mathbf{x})$ is in the lower half of $\mathbf{S}$ and 1 if $\mathbf{H}^i(\mathbf{x})$ is in the upper half of $\mathbf{S}$. Thus the sequence $\{a_i\}$, $i \geq 0$, gives the forward itinerary af $\mathbf{x}$ and $\{a_i\}$, $i \leq 0$, gives the backwards itinerary of $\mathbf{x}$.

Just as the analysis of $\mathbf{H}$ is similar to that of F_c and F_μ, the conclusions drawn are also similar. Smale showed that $\mathbf{H}$ has the following properties:

1) There is a (countably) infinite set of points in Λ_H with periodic orbits.
2) There is an (uncountably) infinite set of points in Λ_H which are not periodic.
3) There is at least one point in Λ_H which has a dense orbit.
4) $\mathbf{H}$ is structurally stable.

Further Reading

A good description, of a somewhat qualitative nature, of the horseshoe map can be found in [Thompson/Stewart]. A more mathematical treatment can be found in both [Devaney] and [Guckenheimer/Holmes].

Problems

1) Find $F'(\mathbf{x})$ and the Jacobian determinant $J_H(\mathbf{x})$ of $\mathbf{H}(\mathbf{x})$ for points located in I_1^-. How will these differ if the horseshoe is constructed with its closed side down rather than up?

2) Show that $d(\mathbf{s},\mathbf{t}) = \Sigma|s_i - t_i|/2^{|i|}$ is a metric on Σ'_2, where the sum is from $-\infty$ to $+\infty$.

3) Let the shift map $\sigma:\Sigma'_2 \to \Sigma'_2$ be defined by $\sigma(\cdots s_{-1}.s_0s_1\cdots) = (\cdots s_{-1}s_0.s_1\cdots)$.

a) Is σ one-to-one? How does this relate to the Jacobian determinant found in #1?

b) Give an example of a point $\mathbf{s}$ in Σ'_2 which has a period three orbit under σ.

Chapter 11 - The Lorenz Equations

In the early 1960's Edward Lorenz was studying meteorology at M.I.T. After graduating from Dartmouth in 1938, Lorenz planned to go into mathematics. Later, at M.I.T., he studied briefly under George Birkhoff. If Poincaré was the grandfather of dynamical systems, Birkoff was the father. However, after the war Lorenz decided to take up meteorology. When he began to notice some unusual and interesting behavior in his computer models, he found himself spending much of his time on mathematics once again.

The set of equations which has come to bear Lorenz' name arises in studies of convection. They are a bare-bone simplification of work dating back to Lord Rayleigh. They are so bare-bones, in fact, that many scientists would look at them and think they couldn't be all **that** bad, that there must be a way to handle the apparently simple nonlinear terms. They were wrong.

Consider a viscous, thermally conducting fluid in a two-dimensional region, a cross-section. Assume the bottom is heated uniformly and that the temperature difference between the top and bottom is constant, ΔT. See Figure 11.1.

```
------------------------------- T2
|                             |
|          fluid:             |
|         viscous,            |      T2 - T1 = ΔT
|         thermally           |
|         conducting          |
|                             |
------------------------------- T1
  ↑  Q  ↑  ↑  ↑  ↑  ↑ Q  ↑
```

Figure 11.1 - Convection Cell

The flow of the fluid and the conduction of heat through it can be modelled by partial differential equations, the unknowns being the velocity vector **v** and the temperature T. The velocity can be represented by a stream function ψ, $\mathbf{v} = (-\partial\psi/\partial y, \partial\psi/\partial x)$. Assuming that ψ and T can be represented by Fourier series, these series in two variables with time dependent coefficients can be substituted into the partial differential equations. The result is a system of ordinary differential equations, one for each time-dependent coefficient. See [Saltzman].

Lorenz decided to look at the dynamics of the first three modes of the Fourier series, or the first three ordinary differential equations out of the infinite set obtained by the procedure described above. These three equations define $\mathbf{L}:\mathbf{R}^3 \rightarrow \mathbf{R}^3$ and are

$$x' = -\sigma(x - y)$$

$$y' = Rx - y - xz$$

$$z' = xy - bz\ .$$

The Prantl number, σ, is a dimensionless number related to the viscous and thermal properties of the fluid. The constant b is geometric in nature, being related to the aspect ratio (a) of the rectangle by $b = 4/(1 + a^2)$. The Rayleigh number, R, is another dimensionless number and is the ratio of driving forces (ΔT) to damping (due to viscous forces and thermal conductivity). All three of these constants are positive.

The first mode, $x(t)$, is the first term in the expansion of ψ. It corresponds to convective circulation in a single large eddy filling the rectangle. Thus the Lorenz system assumes the rectangle is a cross-section of exactly one convective roll. If $x(t)$ is positive, the circulation is clockwise and if $x(t)$ is negative, it is counter-clockwise. The speed is proportional to $|x(t)|$.

The second mode, $y(t)$, describes the horizontal temperature distribution. If $x(t)$ and $y(t)$ have the same sign, the warmer fluid is on the side which is rising. Note that changing the sign of both $x(t)$ and $y(t)$ results in a mirror image flow, with x' and y' changing signs but z' remaining the same.

The final mode, $z(t)$, describes the vertical temperature profile. Although ΔT is constant, the gradient is not necessarily constant across the cell.

When R is small, meaning the driving force is small compared to the damping, the rest state $(0,0,0)$ should be stable. As R increases, the origin should lose its stability as the influx of heat begins to put the fluid into motion. Lord Rayleigh analyzed the onset of convection and predicted it would occur at $R = 1$. Thus the first test of the Lorenz system is whether or not it models this first convective instability.

The derivative of the system is

$$\mathbf{L'} = \begin{bmatrix} -\sigma & \sigma & 0 \\ R-z & -1 & -\sigma \\ y & x & -b \end{bmatrix} .$$

At the origin,

$$\mathbf{L'} = \begin{bmatrix} -\sigma & \sigma & 0 \\ R & -1 & 0 \\ 0 & 0 & -b \end{bmatrix} .$$

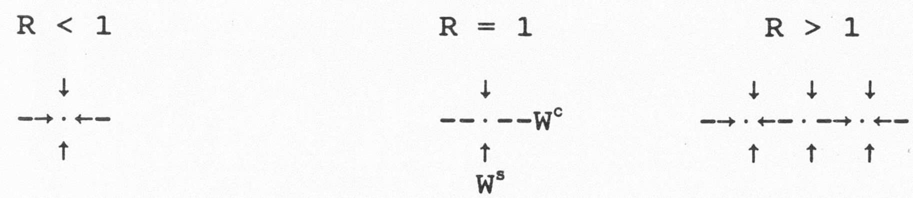
One eigenvalue/eigenvector pair is immediate; $\lambda_3 = -b$, $\mathbf{u}_3 = (0,0,1)$. Thus any small z-mode perturbation from the rest state is damped out.

The other two eigenvalues must satisfy the characteristic equation

$$\lambda^2 - (\sigma + 1)\lambda + (1 - R)\sigma = 0.$$

For $R \approx 0$, the eigenvalues are approximately $-\sigma$ and -1. In this case all eigenvalues are negative, and the origin is an attracting fixed point. At $R = 1$ the eigenvalues at the origin are $-b$, $-(\sigma+1)$, and zero. Thus $R = 1$ is a bifurcation point with a one-dimensional center manifold and a two-dimensional stable manifold. For $R > 1$, the origin is a saddle with index 1. There are now two more real-valued fixed points, $(\pm[b(R-1)]^{1/2}, \pm[b(R-1)]^{1/2}, R-1)$. This type of bifurcation is called a <u>pitchfork bifurcation</u>.

The two new fixed points have real, negative eigenvalues so they are attracting. The eigenvector for the eigenvalue which changed sign at $R = 1$ spans the direction in which the two new points appear, and is tangent to the center manifold of the origin at $R = 1$. Thus the interesting dynamical changes, a reversal of direction for one eigenvector and the appearance of two new critical points, all occurred in the center manifold. A two-dimensional projection of the pitchfork bifurcation is shown in Figure 11.2.



<u>Figure 11.2 - Pitchfork Bifurcation</u>

After the bifurcation at R = 1, the index of the critical point at the origin is 1. Since the co-dimension of the stable manifold for a hyperbolic fixed point equals the index, there is a two-dimensional stable manifold for (0,0,0). This surface divides the phase space into two regions and is called a <u>separator</u>. The two parts of phase space are the <u>basins of attraction</u> for the two attracting fixed points. Trajectories in one basin lead to one of the fixed points while those in the other basin lead to the other fixed point.

At $R = R_h = \sigma(\sigma + b + 3)/(\sigma - b - 1)$ the system undergoes a Hopf bifurcation involving the two fixed points which appeared at R = 1. For $R > R_h$ these points are saddles with index 2, thus the unstable manifolds are two-dimensional.

Lorenz studied the system for the values $\sigma = 10$, $b = 8/3$, and $R = 28$. For these values of σ and b, $R_h \approx 24.74$. This value of σ is too high for dry air and corresponds better to cold water. The value of b was chosen to minimize the value of ΔT for the onset of convection. It has been found that for large values of R seven and fourteen mode truncations display very different behavior, so a three-mode truncation is of questionable physical relevance. Despite this, the Lorenz system has been widely studied by both mathematicians and physicists.

To summarize the critical points for the parameter values chosen by Lorenz, recall that there are three, (0,0,0) and the mirror-image points $(\pm[b(R-1)]^{1/2}, \pm[b(R-1)]^{1/2}, R-1)$. These last two are now some distance from the origin. All three are saddles of index 2. The linearized system for each one has one negative, real eigenvalue and a complex conjugate pair with positive real part. Trajectories of points nearby are drawn towards these fixed points along their one-dimensional stable manifolds and then move away, spiralling, along the unstable manifold surface. See Figure 11.3.

While there is no final steady-state flow, it is possible to find a region of phase space enclosing all three fixed points such that no trajectory leaves the region. This indicates that all final motions are bounded. Actually, a nested sequence of such regions can be found.

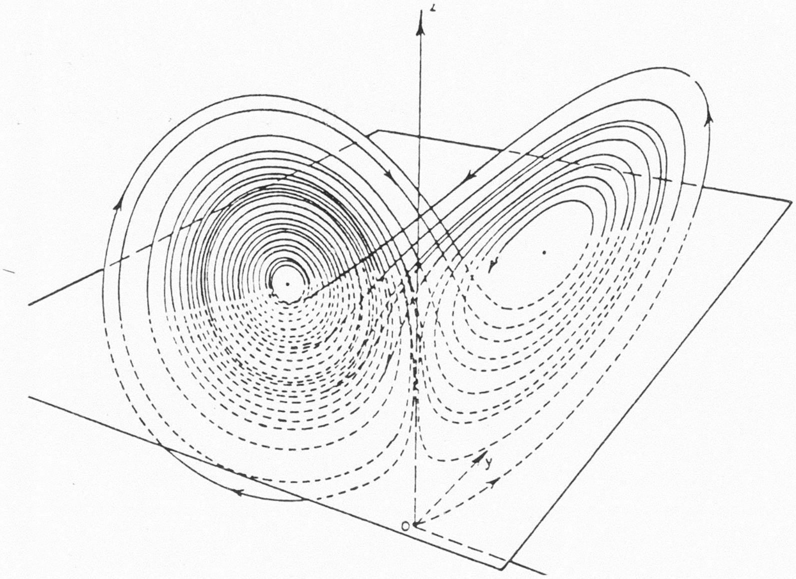


Figure 11.3 - Lorenz System

To see why this 'shrinking' may be expected, the transport theorem,

$$d/dt \quad _{vol} F\, d^3x = \quad _{vol} (dF/dt + F \cdot \mathbf{v})\, d^3x \ ,$$

may be applied to phase space. Let $F = 1$ and let $\mathbf{v} = \mathbf{L}$. Then

$$d(vol)/dt = \quad _{vol} \operatorname{div} (\mathbf{L})\, d^3x.$$

The divergence of the Lorenz system, div $\mathbf{L}$, is $\partial x'/\partial x + \partial y'/\partial y + \partial z'/\partial z = -(\sigma + b + 1) < 0$. Thus the phase space shrinks with time to some set with zero volume, such as a surface.

In closing, some additional conclusions should be noted although they will not be justified mathematically here. While the global behavior can only be investigated numerically, it has been found that a trajectory will spiral out from one of the two saddles other than the origin. After some number of turns, which is unpredictable, it is 'captured' by the other saddle. At this point it approaches that saddle then spirals outwards from it. The

ω-limit set turns out to be a 'surface' comprised of layers such that a cross-section of it is a cantor-like-set. Thus it has zero volume. Finally, the Lorenz system is <u>never</u> structurally stable, for any value of R!

The Lorenz system has been widely studied and much has been written about it. Many studies have examined the behavior as a function of R, holding σ and b constant, as has been done here. However, it has been examined as a function of σ also. It has also been established that as $R \to \infty$, the system has steady, periodic solutions.

Further Reading

Both [Thompson/Stewart] and [Guckenheimer/Holmes] devote a considerable amount of space to the Lorenz system. The second is very technical, the first is not. Many references to other works are contained in both. Anyone interested in chaotic dynamics should also read the two classic papers by [Lorenz].

Appendix A - Review Problems

These problems are intended to serve as a review and to give some practice at basic skills. They can be considered to be a sort of prerequisite in that the reader is expected to be able to do them without much trouble, although some review may be required.

1) The purpose of this exercise is to give you a feel for an apparently simple map which we will be studied in detail.

Let $x_{n+1} = f(x_n) = x_n^2 + c$.

Using a hand calculator or a simple computer program, iterate this for three different values of c; $c > 1/4$, $c = 1/4$, and $c < 1/4$. Choose several arbitrary initial values for each case. For each value of c, sketch a graph of $y = f(x_n)$. Include the line y=x on each of the graphs.

2) One of the most basic biological models used for single species populations is the 'logistic equation':

$$dP/dt = kP(1-P)$$

where P is the population (normalized in relation to carrying capacity). [See, for example, "Wildlife Ecology and Management" by Robinson/Bolen.]

i) Describe what happens when $P = 1$, $P < 1$, and $P > 1$.

ii) Under what conditions will the population grow exponentially? (Hint: What equation would have an exponential solution $P=P_0e^{kt}$?) When might this occur?

iii) Solve the differential equation by separation of variables and integration by partial fractions.

3) The goal of this exercise is to find a solution to the dynamical system represented by:

$$x_1' = 5x_1 + 3x_2 \qquad x_1(0) = a$$

$$x_2' = -6x_1 - 4x_2 \qquad x_2(0) = b$$

where x_1, x_2 are functions of time t and ' indicates a time derivative.

i) Put the system in matrix form, $\mathbf{x}' = A\mathbf{x}$.

ii) Find the eigenvalues and the associated eigenvectors of the 2x2 matrix A.

iii) Let Q be a matrix with the eigenvectors of A as columns. Find Q^{-1} and verify that $B=Q^{-1}AQ$ is diagonal.

iv) Let $\mathbf{x} = Q\mathbf{y}$ and rewrite the system in the form $\mathbf{y}' = B\mathbf{y}$. Find y_1, y_2 in terms of x_1, x_2 and the initial conditions for $\mathbf{y}$ in terms of those given for $\mathbf{x}$.

v) Solve the (uncoupled) system for y_1 and y_2.

vi) Find $x_1(t)$ and $x_2(t)$ from y_1, y_2.

vii) Verify that these solutions satisfy the original system of equations and initial conditions.

Appendix B - Problem Solutions

<u>Review Problems (Appendix A)</u>

2) i) If $P = 1$, $dP/dt = 0$ and the population is in equilibrium.

If $P < 1$, $dP/dt > 0$ and the population will grow.

If $P > 1$, $dP/dt < 0$ and the population will shrink.

ii) $dP/dt = kP$ would result in exponential growth. This would hold approximately for $P << 1$. An example would be the introduction of a new species to favorable habitat with negligible predation and/or disease.

iii) $dP/P(1-P) = k\,dt$ (separate variables)

$dP/P - (-dP)/(1-P) = k\,dt$ (partial fractions)

$\ln P - \ln(1-P) = kt + C$ (integrate)

$\ln(P/1-P) = kt + C$ (simplify logs)

$P = ce^{kt}/(1 + ce^{kt})$ (solve for P)

$P(0) = P_0 = c/1+c$ (initial population)

$c = P_0/1-P_0$ (find c from i.c.)

$P = P_o e^{kt}/1 - P_o + P_o e^{kt}$ (solution)

3) i) $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ -6 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

ii) $\begin{vmatrix} 5-\text{☺} & 3 \\ -6 & -4-\text{☺} \end{vmatrix} = 0$ $\quad$ ☺ = 2, -1

☺ = 2: $\begin{bmatrix} 3 & 3 \\ -6 & -6 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad w_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

☺ = -1: $\begin{bmatrix} 6 & 3 \\ -6 & -3 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad w_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

iii) $Q = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$ $\quad Q^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

$B = Q^{-1}AQ = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$

iv) $x = Qy$; $y = Q^{-1}x$

$Q^{-1}x' = Q^{-1}Ax = (Q^{-1}AQ)Q^{-1}x$

$y' = By$

$y_1 = 2x_1 + x_2$
$y_2 = x_1 + x_2$

$y_1(0) = (2a + b)$
$y_2(0) = (a + b)$

v) $y_1' = 2y_1 \quad y_1 = A\, e^{2t} = (2a + b)e^{2t}$
$y_2' = -y_2 \quad y_2 = B\, e^{-t} = (a + b)e^{-t}$

vi) $x_1 = y_1 - y_2$
$x_2 = -y_1 + 2y_2$

$x_1 = (2a + b)e^{2t} - (a + b)e^{-t}$
$x_2 = -(2a + b)e^{2t} + 2(a + b)e^{-t}$

vii) $x_1' = 2(2a + b)e^{2t} - (a + b)e^{-t} = 5x_1 + 3x_2$
$x_2' = -2(2a + b)e^{2t} - 2(a + b)e^{-t} = -6x_1 - 4x_2$

$x_1(0) = 2a + b - a - b = a$
$x_2(0) = -2a - b + 2a + 2b = b$

Chapter 1 Problems

1) Free Body Diagram →

$\Sigma F_x = mg \sin\theta = m\ell\theta''$

$\ell\theta'' - g \sin\theta = 0$

$\theta'' - (g/\ell) \sin\theta = 0$

Linearize:

$\sin\theta = \theta + \text{h.o.t.}$ (Taylor Series)

$\sin\theta \approx \theta$ for small θ

$\theta'' - (g/\ell)\,\theta = 0$

2) (i) If $\alpha_1 = \alpha_2 = 0$ then each populaton is described by the logistic equation. (They are uncoupled.)

(ii) Three can be found from inspection of the equations: $(0,0)$ $(0,K_2)$ $(K_1,0)$.

The fourth equilibrium point is found by solving:

$$0 = K_1 - x - \alpha_1 y$$
$$0 = K_2 - y - \alpha_2 x$$

The solutions are:

$$x_o = (K_1 - \alpha_1 K_2)/(1 - \alpha_1\alpha_2)$$
$$y_o = (K_2 - \alpha_2 K_1)/(1 - \alpha_1\alpha_2)$$

(iii) $(x_o + \delta x)' = \beta_1(x_0 + \delta x)[K_1 - (x_o + \delta x) - \alpha_1(y_o + \delta y)]$

$$\delta x' = \beta_1 x_0[K_1 - (x_o + \delta x) - \alpha_1(y_o + \delta y)] + \beta_1\delta x[K_1 - (x_o + \delta x) - \alpha_1(y_o + \delta y)]$$

$$\delta x' = \beta_1 x_o(K_1 - x_o - \alpha_1 y_o) + \beta_1 x_o(-\delta x - \alpha_1\delta y) + \beta_1 y_o\delta x - \beta_1 K_1\delta x - \beta_1\alpha_1 y_o\delta x$$

The first term is zero (equilibrium condition) so:

$$\delta x' = \beta_1(K_1 - x_o - \alpha_1 y_o)\delta x - \beta_1 x_o\delta x - \beta_1\alpha_1 x_o\delta y$$

$$\delta x' = -(\beta_1 x_o)\delta x - (\beta_1\alpha_1 x_o)\delta y$$

Similarly:

$$\delta y' = -(\beta_2\alpha_2 y_o)\delta x - (\beta_2 y_o)\delta y$$

The linearized system is:

$$\begin{bmatrix} \delta x \\ \delta y \end{bmatrix}' = \begin{bmatrix} -\beta_1 x_o & -\beta_1 \alpha_1 x_o \\ -\beta_2 \alpha_2 y_o & -\beta_2 y_o \end{bmatrix} \begin{bmatrix} \delta x \\ \delta y \end{bmatrix}$$

3) The trajectory appears to cross itself because the view is actually a projection of the full phase space. We could add a time axis into the page. The trajectory would not cross itself in the three dimensional space (x,x',t).

Chapter 2 Problems

1) d_1: The interior of the unit circle
d_2: A square with sides of length 2 centered at origin
d_3: A diamond centered at origin with (0,1) and (1,0) as two of its vertices.

2) Let $\epsilon = 1/2$. Let $p \in \mathbf{Q}$. For any δ we can find $x \notin \mathbf{Q}$ such that $|p-x| < \delta$, but $|f(p) - f(x)| > \epsilon$. Likewise, for $p \notin \mathbf{Q}$, but in this case $x \in \mathbf{Q}$. (That such an x can be found follows from the rationals and irrationals being dense in [0,1]).

3) (i) $\sinh^{-1} x = \ln[x + (x^2 + 1)^{1/2}]$ (from tables)
$(\sinh^{-1} x)' = (1 + x^2)^{1/2}$ (also from tables)

sinh x is 1-1, onto, and continuous and its inverse is continuous, so it is a homeomorphism.

Since sinh x and its inverse are both C^1, sinh is a C^1-diffeomorphism .

(ii) cosh x is not 1-1, so it is not a homeomorphism.

(iii) x^n is 1-1, onto and continuous if n is odd.
The inverse, $x^{1/n}$, is not C^1 (at 0) so x^n is not a diffeomorphism.

(iv) $J_o(x)$ is oscillatory, so it is not 1-1.

4) (i) 0

(ii) $1/n$, $n \in \mathbf{Z}^+$

(iii) 2^{-n} , 5^{-n} $n \in \mathbf{Z}^+$

5) Area after the nth step:

n=1: $(2/5)^2 \cdot (4) = (4/5)^2$
n=2: $(4/25)^2 \cdot (16) = (4/5)^4$
n=3: $(4/5)^6$

n=i: $(4/5)^{2i}$

$\lim_{n\to\infty}$ (Area) = 0

Since the area remaining is zero, there are no two-dimensional intervals remaining. Since any horizontal or vertical 'cut' is a middle-fifths cantor set, there are no one-dimensional intervals remaining either.

That the set is closed and that all points are limit points can be shown in a manner analogous to that used for Λ_3. To construct the sequences in this case, corners can be used.

6) i) $d(\mathbf{s},\mathbf{s}) = \Sigma(0/2^i) = 0$

$d(\mathbf{s},\mathbf{t}) > 0$ if $\mathbf{s} \neq \mathbf{t}$ since $s_i \neq t_i$ for some i and since every nonzero term in the sum is positive

$d(\mathbf{s},\mathbf{t}) = d(\mathbf{t},\mathbf{s})$ since $|s_i - t_i| = |t_i - s_i|$

$d(\mathbf{s},\mathbf{r}) + d(\mathbf{s},\mathbf{t}) \geq d(\mathbf{r},\mathbf{t})$ since $|r_i-s_i| + |s_i-t_i| \geq |r_i-t_i|$

ii) $d(\mathbf{s}_1,\mathbf{s}_2) = [\Sigma(1/2)^i] - 1 = [1-(1/2)]^{-1/2} - 1 = 2 - 1 = 1$

$d(\mathbf{s}_1,\mathbf{s}_3) = [\Sigma(1/2)^i] - 1 - 1/2 - 1/4 = 1 - 3/4 = 1/4$

$d(\mathbf{s}_1,\mathbf{s}_4) = [\Sigma(1/2)^i] - 1 - 1/2 - 1/4 - 1/8 = 1/4 - 1/8 = 1/8$

'Close' means agreeing in the first terms

iii) $M = \Sigma(1/2)^i = 2$

$\mathbf{s} = (000 \ldots)$ $\mathbf{t} = (111 \ldots)$

$\mathbf{s} = (0101 \ldots)$ $\mathbf{t} = (1010 \ldots)$

iv) Let $\mathbf{t} = (t_1t_2t_3 \ldots)$. Let $\tau_n = (t_1\ldots t_n\ t_1\ldots t_n \ldots)$. Thus τ_n agrees with $\mathbf{t}$ in the first n places, then repeats. As $n\to\infty$ $\tau_n \to \mathbf{t}$.

Chapter 3 Problems

1) $F_{-1}(x) = x^2 - 1$
$F'(x) = 2x$

Fixed Points: $x = x^2 - 1$
$x^2 - x - 1 = 0 \Rightarrow x = (1 \pm \sqrt{5})/2$

F' at fixed points: $(1 \pm \sqrt{5}) \Rightarrow$ both repelling

Prime Period 2 pts: $-1, 0$

$$\begin{aligned}(F^2)' &= F'(F(x)) \cdot F'(x) \quad \text{(Chain Rule)}\\ &= 2F(x) \cdot 2x\\ &= 4x(x^2 - 1)\end{aligned}$$

$|(F^2)'(0)| = |(F^2)'(-1)| = 0 < 1 \Rightarrow$ attracting

2) Fixed points: $x = x^2 + c$
$x^2 - x + c = 0$

$x = (1 \pm \sqrt{1 - 4c})/2$ (two distinct real solutions if $c < 1/4$.)

Derivative at fixed points: $1 \pm (\sqrt{1 - 4c})$

The larger (+) is greater than one for $c < 1/4$.
The smaller one is less than 1 in magnitude for $c > -3/4$.

Chapter 4 Problems

1) Property 1: $Per_n(\sigma)$ has 2^n elements; they are the sequences which repeat every n entries, or the 2^n binary numbers with n digits.

Property 2: Follows immediately from #6(iv) in Chapter 3.

Property 3: $\mathbf{s} = (0\ 1\ 00\ 01\ 10\ 11\ 000\ 001\ 010\ 011\ 100\ 101\ldots)$. Choose any point $\mathbf{t}$. For any i $\sigma^n(\mathbf{s})$ will equal $\mathbf{t}$ in the first i places for some n.

2) No. It is not 1-1 (it's 2-1) and it has eventually fixed points such as (011011111...). See Theorem 3.1.

3) Let $f = x^2 + c$ and $g = \mu x(1 - x)$ in the definition of a topological conjugacy. Then

$$h \circ f = \alpha(x^2 + c) + \beta \quad \text{and}$$

$$g \circ h = \mu(\alpha x + \beta)(1 - \alpha x - \beta).$$

Equating these, expanding, and then equating the coefficients of x^2, x^1, x^0 on each side gives

$$\alpha = -(\mu\alpha^2) \text{ or } \mu = -1/\alpha\ ,$$

$$\mu\alpha(1 - 2\beta) = 0 \text{ so } \beta = 1/2\ , \text{ and}$$

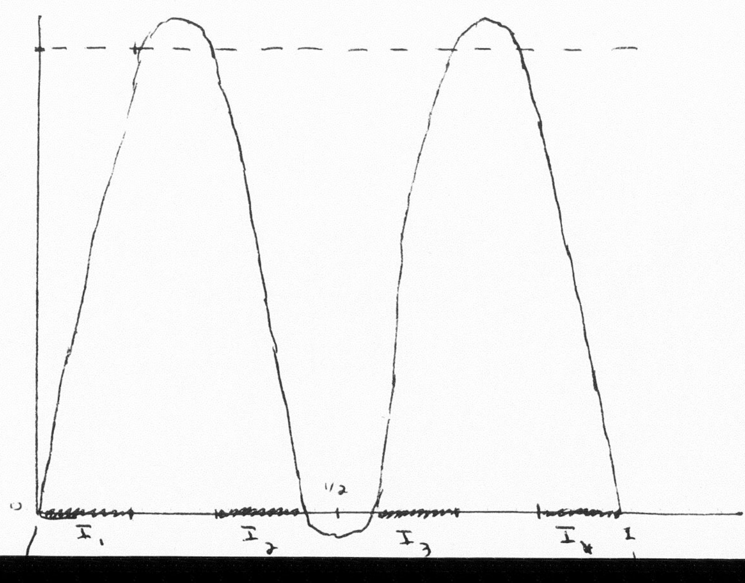
$$\alpha c + \beta = \mu\beta(1 - \beta).$$

Solving the last equation for μ in terms of c,

$$\mu = 1 \pm \sqrt{1-4c}.$$

Since h(x) is the equation of a line it is C^0, 1-1, and onto and is therefore a homeomorphism.

4)



In each of I_1, I_2, I_3, and I_4 three subintervals are mapped into components of A_0, and therefore escape. This process continues, so that the set of points with bounded orbits is a Cantor set.

The appropriate conjugate space would be Σ_4.

The conjugacy would be $f(x)=\{s_0 s_1 \cdots\}$ where $s_n = i$ if $f^n(x) \in I_i$.

The properties would be:

1) $Per_n(\sigma)$ has 4^n elements
2) $Per(\sigma)$ is dense in Σ_4
3) σ has a dense orbit in Σ_4.

A dense orbit would be:

$\mathbf{s}$ = (0 1 2 3 00 01 02 03 10 11 12 13 20 21 22 23 30 31 32 33 000 001 $\cdots$).

Note that this is what we <u>expect</u> to work. Nothing done here constitutes a rigorous proof of anything.

Chapter 5 Problems

1) $T_\lambda(\theta) = \theta + 2\pi(p/q)$; $T_\lambda^q(\theta) = \theta + 2\pi p = \theta$.

T_λ is not chaotic in either case. It does not have sensitive dependence on initial conditions. Also, if $\lambda \in \mathbf{Q}$ it is not topologically transitive and if $\lambda \notin \mathbf{Q}$ there are no periodic points.

2)
$$y = \cos(n \arccos x)$$
$$y' = n \sin(n \arccos x)/(1 - x^2)^{1/2}$$
$$y'' = nx \sin(n \arccos x)/(1-x^2)^{3/2} - n^2 \cos(n \arccos x)/(1 - x^2)$$

Substituting these into the differential equation shows that it is satisfied.

3) Let $\theta_p \in Per_i(g_n)$.

$T_n^i(\cos \theta_p) = \cos (n^i \arccos (\cos \theta_p)) = \cos n^i\theta_p$.

Since $\theta_p \in Per_i(g_n)$, $n^i\theta_p = \theta_p$, so $T_n^i(\cos \theta_p) = \cos \theta_p$.

4) If $b = 0$, $x_{n+1} = 1 - ax_n^2$. Let $\bar{t} = -ax$, or $x = -\bar{t}/a$.

Then $-\bar{t}_{n+1}/a = 1 - \bar{t}_n^2/a$, or $\bar{t}_{n+1} = t_n^2 - a$.

5) a) $T_3(x) = 4x^3 - 3x$, $T_2(x) = 2x^2 - 1$

$T_4(x) = 2x(4x^3 - 3x) - 2x^2 +1 = 8x^4 - 8x^2 + 1$

b) $h \circ g = \cos 4\theta$

$T \circ h = 8\cos^4 \theta - 8\cos^2 \theta + 1$

$\cos 4\theta = 8\cos^4 \theta - 8\cos^2 \theta + 1$ (multiple angle formula)

6) $H_2(x) = 2x(2x) - 2(1)(1) = 4x^2 - 2$

H_2 is conjugate to $F_{-8}(x) = x^2 - 8$ through the homeomorphism $f(x) = 4x$. Thus the dynamics are equivalent.

Chapter 6 Problems

1) a) $f(x)$ oscillates between the curves $x/2 \pm x^2$. The closer it gets to zero, the more it oscillates per unit length. In any neighborhood of the origin, it has infinitely many local extrema. Thus it is not one-to-one on any interval $(-\epsilon,\epsilon)$.

b) $f'(0) = \lim_{h\to 0} (1/h)[f(h) - f(0)] = \lim_{h\to 0}[(h/2) + h^2\sin(1/h)]/h = \lim_{h\to 0} [1/2 + h \sin(1/h)] = 1/2$. This does not contradict the implicit function theorem because $f(x)$ and $f'(x)$ are not continuous at $x = 0$.

2) $F(e^{-2},2) = e^{-2}e^2 - 2 + 1 = 0$. $\partial F/\partial x = e^y$, $\partial F/\partial x(e^{-2},2) = e^2 \neq 0$. Therefore, $x = f(y)$ in a neighborhood of $(e^{-2},2)$. However, $\partial F/\partial y = xe^y - 1$, $\partial F/\partial y(e^{-2},2) = 0$, so the implicit function theorem does not apply in this case.

From implicitly differentiating $F(x,y)$ with respect to y, it is found that $x' = e^{-y} - x$, and $x'' = -e^{-y} - 1$. Since $x'(e^{-2},2) = 0$ and $x''(e^{-2},2) < 0$, $x = f(y)$ has a minimum at $(e^{-2},2)$. Therefore $x = f(y)$ is not one-to-one and cannot be inverted. Thus y cannot be written as a function of x at this point.

3) $F(\mu,x) = \mu x(1 - x)$.

a) $F' = [x(1 - x), \mu(1 - 2x)] = [\partial F/\partial\mu, \partial F/\partial x]$.

b) $\mu = f(x)$ for $x \neq 0,1$ and $F(\mu,x) = 0$.
$x = g(\mu)$ for $\mu \neq 0$, $x \neq 1/2$ and $F(\mu,x) = 0$.
Note that for $x = 1/2$ and $F = 0$ $\mu = 0$, so this is actually one case.

c) From implicit differentiation:

$\mu' = \mu(2x - 1)/x(1 - x)$, so $\mu'(0,x) = \mu'(0,1/2) = 0$.

From inspection of the equation for μ', it can be determined that μ' changes sign as μ passes through 0. Thus $\mu = f(x)$ is not one-to-one at those points, so $x = g(\mu)$ cannot be defined there.

$x' = x(x -1)/\mu(1 - 2x)$, so $x'(\mu,0) = x'(\mu,1) = 0$.

From a similar argument, it is established that $\mu = f(x)$ cannot be defined for these conditions.

4) T must be invertible, or $\det T \neq 0$.

5) They are solutions to the characteristic polynomial, which is a polynomial with real coefficients. It could be written as $p(x) = \prod_i(x - \lambda_i)$. If some λ_i is complex, its conjugate must appear as some λ_j in order for $p(x)$ to have real coefficients.

6) $p_B = \det(\lambda I - B)$
$= \det(\lambda T^{-1}T - T^{-1}AT)$
$= \det(T^{-1}(\lambda I - A)T)$
$= (\det T^{-1})[\det(\lambda I - A)](\det T)$
$= \det(\lambda I - A) = p_A$

7) a) $r^2 = x^2 + y^2$
$2rr' = 2xx' + 2yy'$ or $rr' = xx' + yy'$
write x' and y' in terms of r and θ, multiply by $(r \cos \theta)$ and $(r \sin \theta)$ respectively, add, and divide by r to obtain:

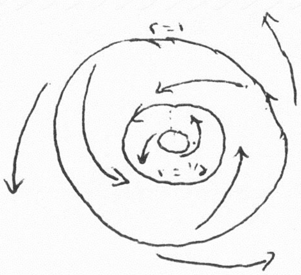
$r' = r^3 \sin(\pi/r)$

$y = r \sin \theta$
$y' = r' \sin \theta + r(\cos \theta)\theta'$, or $\theta' = (y' - r' \sin \theta)/r \cos \theta$
use y' in terms of r and θ as above, and use the expression above for r', to get:

$\theta' = 1$

b) If $2n < 1/r < 2n + 1$, $r' > 0$ (i.e. if $1/2n > r > 1/2n+1$)
If $2n > 1/r > 2n - 1$, $r' < 0$ (i.e. if $1/2n < r < 1/2n-1$)

c)



8) $(0,0)$: $\lambda_1 = \beta_1 K_1$, $\lambda_2 = \beta_2 K_2$; repelling node

$(0,K_2)$: $\lambda_1 = -\beta_2 K_2$ (attracts)
$\lambda_2 = \beta_1(K_1 - \lambda_1 K_2)$ (depends on $K_1 - \lambda_1 K_2$)

$(K_1,0)$: $\lambda_1 = -\beta_1 K_1$ (attracts)
$\lambda_2 = \beta_2(K_2 - \lambda_2 K_1)$ (depends on $K_2 - \lambda_2 K_1$)

(x_0,y_0): $\lambda_{1,2} = 1/2\{[-(\beta_1 x_0 + \beta_2 y_0)] \pm [(\beta_1 x_0 + \beta_2 y_0)^2 - 4(1-\lambda_1\lambda_2)\beta_1\beta_2 x_0 y_0]^{1/2}\}$

When (x_0,y_0) is in the first quadrant, there are two cases:

i) $\lambda_1\lambda_2 > 1$: (x_0,y_0) is a saddle, other two are attractors

ii) $\lambda_1\lambda_2 < 1$: (x_0,y_0) is an attractor, others are saddles

Notes: Remember that β_i and K_i are intrinsically positive, since they represent carrying capacities and growth rates. Also recall that populations are positive, so only the first quadrant is of interest.

Chapter 7 Problems

1) If two stable manifolds of distinct critical points intersect, then the solution to the system is not unique for each **x**. If two unstable manifolds intersect, the system is not deterministic.

2) Each trajectory, or any union of trajectories, is an invariant set. The nonwandering set Ω contains only the origin. Since the origin is a fixed point, it is a trajectory. Thus Ω contains one and only one invariant set.

3) There is one. For $\mu = 4$ it is the interval $[0,1]$ and for $\mu > 4$ it is $\Lambda_\mu \subset [0,1]$.

4) For $\{\mathbf{x} | r > 1\}$, the α-limit set is $\{\mathbf{x} | r = 1\}$, and the ω-limit set is empty. For each circle, $S_n = \{\mathbf{x} | r = 1/n\}$, the α-limit set and the ω-limit set is itself, S_n. For any set of points in an annular region between two circles S_n and S_{n+1}, the circle with odd index is the α-limit set and the one with even index is the ω-limit set.

Chapter 8 Problems

1) a) $V = -xy$
$x' = -\partial V/\partial x$
$y' = -\partial V/\partial y$

fixed points: (0,0)

All fixed points are hyperbolic, so this system is structurally stable.

b) $(0,\pm 1)$ are fixed points, and are saddles, since

$$F'(0,\pm 1) = \begin{bmatrix} \pm(-3) & 0 \\ 0 & \pm 2 \end{bmatrix}.$$

When $x = 0$, $x' = 0$ and $y' = -1 + y^2$.

For $|y| < 1$, $y' < 1$, so there is a trajectory connecting the two saddles. Therefore, this system is structurally unstable.

c) $V = -(1/2)\exp(x^2 + y^2)$
$x' = -\partial V/\partial x$
$y' = -\partial V/\partial y$

The only fixed point is (0,0) and $F'(0,0) = I$, so the fixed point is hyperbolic. Thus this system is structurally stable.

d) Chapter 6, prob. 7 - This system does not have a finite number of limit cycles, so the system is structurally unstable.

2) $T_\lambda : S^1 \to S^1$, $T_\lambda(\theta) = \theta + 2\pi\lambda$.

Chapter 9 Problems

1) a) Let $G(x,\lambda) = f_\lambda(x) - x$. If $G(x,\lambda) = 0$, then f_λ has a fixed point at x. By hypothesis, $G(x_0,\lambda_0) = 0$. $\partial G/\partial\lambda = \partial f_\lambda/\partial\lambda$, and at $\lambda = \lambda_0$ this is nonzero by hypothesis. Since $\partial G/\partial\lambda|_{\lambda=\lambda_0} \neq 0$, the implicit function theorem gives a smooth function $\lambda = p(x)$ such that $G(x,p(x)) = 0$, or $f_{p(x)}(x) = x$.

b) $\partial G/\partial x + [\partial G/\partial\lambda]p'(x) = 0$, so $p'(x) = -(\partial G/\partial x)/(\partial G/\partial\lambda)$. $\partial G/\partial x = f_\lambda'(x) - 1$. At (x_0,λ_0), $f_{\lambda 0}'(x_0) = 1$ by hypothesis, so $p'(x_0) = 0$.

c) Write $p'(x) = -f(x,y)/g(x,y)$. Then use the quotient rule to get $p''(x) = \{-g(x,y)[f_x(x,y) + f_y(x,y)y_x] + f(x,y)[\ldots]\}/[g(x,y)]^2$. At x_0, note that $y_x = p'(x) = 0$ and $f(x,y) = \partial G/\partial x = 0$. Therefore, $p''(x) = -g(x,y)f_x(x,y)/[g(x,y)]^2 = -(\partial^2 G/\partial x^2)(\partial G/\partial\lambda)/(\partial G/\partial\lambda)^2 \neq 0$.

2) a) Critical point: (0,0). (This is a linear system.)

$$F'(0,0) = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \Rightarrow \text{hyperbolic.}$$

b) $dy/dx = (b/a)y/x$
$y = C|x|^{b/a}$
$y' = (b/a)C|x|^{(b/a)-1} = (b/a)y/x$

c) If $r < b/a < r+1$, $y \in C^r$ but $y \notin C^{r+1}$.

3) a) Critical points: $(0,0,\mu),(c,0,c^2)$ for $c > 0$.

$$F'(\mathbf{x}) = \begin{bmatrix} \mu-3x^2 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} .$$

(i) $$F'(0,0,0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{non-hyperbolic}$$

(ii) $$F'(0,0,\mu) = \begin{bmatrix} \mu & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{hyp. in } \mu\text{=const plane}$$

(iii) $$F'(\pm\surd c,0,c) = \begin{bmatrix} -2c & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{hyp.in } \mu\text{=const plane}$$

b) (i) E^u = y-axis; E^c = x-μ plane
(ii) repelling node for $\mu > 0$; saddle for $\mu < 0$
(iii) saddle

c) for (0,0,0): $W^s = \emptyset$
W^u = y-axis
W^c = x-μ plane

Chapter 10 Problems

1) $H' = \begin{bmatrix} \pm\lambda & 0 \\ 0 & \pm\mu \end{bmatrix}$, $J_H = \lambda\mu$.

The + signs apply to the part which is not flipped over.

2) Clearly, $d(\mathbf{s},\mathbf{t}) \geq 0 \ \forall \ \mathbf{s},\mathbf{t}$ and $d(\mathbf{s},\mathbf{t}) = 0$ iff $s_i = t_i \ \forall \ i$. Since $|s_i - t_i| = |t_i - s_i|$, $d(\mathbf{s},\mathbf{t}) = d(\mathbf{t},\mathbf{s})$. If $\mathbf{r},\mathbf{s},\mathbf{t} \in \Sigma'_2$, then $|r_i - s_i| + |s_i - t_i| \geq |r_i - t_i|$, so $d(\mathbf{r},\mathbf{s}) + d(\mathbf{s},\mathbf{t}) \geq d(\mathbf{r},\mathbf{t})$.

3) a) σ is one-to-one. Since σ is topologically conjugate to H, they are both invertible. The inverse function theorem requires $J_H \neq 0$ for H to be invertible.

b) $\cdots 01101101.1011011 \cdots \in Per_3(\sigma)$.

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